Soft threshold stochastic resonance

Priscilla E. Greenwood
Arizona State University

Ursula U. Müller
Universität Bremen

Lawrence M. Ward
University of British Columbia

Abstract

Soft thresholds are ubiquitous in living organisms, in particular in mechanisms of neurons and of neural networks such as sensory systems. Which soft threshold functions produce (threshold) stochastic resonance remains a question. The answer may depend on the information measure used. We argue that Fisher information about signal parameters is an attractive measure of information transmission across soft thresholds. We illustrate how the pattern of information changes as a signal moves across a soft threshold. For some signals this pattern is much the same whether Fisher information or signal-to-noise ratio is used as a measure of information transmission. Non-invertibility of the threshold function, rather than its steepness, is important for stochastic resonance measured by Fisher information.

PACS number(s): 87.10.+e, 05.45.-a, 02.50.Ph, 05.40.Ca

1 Introduction

Suppose we know whether a noisy signal has crossed a threshold in each of a sequence of trials. The data can be represented as a series of zeros and ones corresponding to signal plus noise being below or above the 'sharp' threshold. Information about a subthreshold signal increases as the standard deviation of the noise, σ, increases from 0 and then decreases as σ increases further. We call the function that relates the amount of information about a signal transmitted across a sharp threshold as a function of the noise standard deviation, σ, the *information transmission function*. The value of σ at which the information transmission function reaches a maximum is called the *stochastic resonance point* and the phenomenon is called *threshold stochastic resonance*. Dynamical and threshold stochastic resonance have been well-studied in many physical systems such as the earth's climate (Benzi et al., 1981) and in a variety of information processing systems, including living ones such as neuron firing models and networks of neurons (e.g. Collins et al., 1996; Longtin, 1993, 1997; Gluckman et al., 1996). For reviews see Gammaitoni et al. (1998), Anishchenko et al. (2002) and Ward (2002). Threshold stochastic resonance has been explored statistically (e.g. Greenwood et al., 1999; Müller, 2000; Müller & Ward, 2000).

It is well-known that biological systems under ordinary conditions usually do not exhibit Heaviside-type threshold functions, but rather respond to weak signals in a graded way over some range (e.g. Geldard, 1972). In human and animal psychophysics, the relationship between an organism's sensitivity to a sensory stimulus and the intensity of the stimulus is called the "psychometric function". Psychometric functions are usually modeled by one of three similar functions, depending on theoretical considerations and goodness of fit: the logistic function, the Gaussian distribution function or the Weibull function (Macmillan & Creelman, 1990). All can be considered to be soft threshold transfer functions because they all describe a monotonic increase in transmitted information about a weak sensory signal as a function of signal intensity, with the steepness of the function determined by a parameter. One or another of these three functions can be said to describe the vast majority of sensory and neural thresholds, almost always with steepness parameter values that make them quite different from the discontinuous Heaviside function. It is important to understand when and how stochastic resonance appears in such soft threshold systems.

Suppose that, instead of having a sharp threshold response represented by zeros and ones, a system possesses a soft threshold, i.e. for some nondecreasing function with values going from 0 to 1 we see the function of the noisy signal. Does the system still exhibit stochastic resonance? A number of papers identify circumstances where the answer is yes. In particular, Chapeau-Blondeau and Godivier (1997) showed that stochastic resonance is obtained with a periodic signal and, for example, a logistic soft threshold transfer function, using signal-to-noise ratio as a measure of information transmission. Vilar et al. (1998), using a similar signal-to-noise ratio measure, provided examples of soft threshold transfer functions that yield stochastic resonance and suggested a general criterion for which transfer functions would have this property.

The choice of measurement of information transmission, important for the study of stochastic resonance with a sharp threshold, becomes even more important when the threshold is soft. Ward et al. (2002) explored the significance of a number of measures of information transmission including the Fisher information lower bound suggested by Stemmler (1996). For periodic signals they used a version of signal-to-noise ratio and a measure from signal detection theory. Rousseau et al. (2003) studied sensors with saturation, another formulation of soft threshold, using various measures of information transmission depending on the input signal type. Ward (2003) showed that the d' measure from signal detection theory exhibits stochastic resonance for the logistic function and for the exponential function.

In this paper we investigate the stochastic resonance phenomenon with a variety of soft threshold transfer functions, using Fisher information as a measure of information transmission.

We introduce the Fisher information for a periodic signal in discrete time which is a weighted sum of Fisher information values for a set of constant signals that approximate the periodic signal. This enables us to compare results for periodic signals using Fisher information with results using signal-to-noise ratio.

Suppose the transfer function increases only in an interval, which we call a window, and is 0 below the window and 1 above it. We show that Fisher information does not depend on the shape of the soft threshold, but only on the position of the signal relative to the window where part of the noisy signal is observed. This result greatly simplifies the study, reducing it to the exploration of how the stochastic resonance phenomenon depends on the window itself. We study this question. Computations indicate that if the signal is inside the soft threshold window, rather than outside it, there is no stochastic resonance. In other words, the edges of the window are critical points for the emergence of stochastic resonance.

In Section 2, we review the general property of Fisher information that it is unchanged by any invertible transformation of a constant, periodic or aperiodic signal. This implies that stochastic resonance cannot appear if the measure is Fisher information and if the soft threshold function is invertible.

In Section 3, we define a large and convenient family of non-invertible soft threshold functions in terms of non-invertible segments above and below a monotonic, invertible function, h, and compute for a constant signal, s, the Fisher information about s. We see that the Fisher information does not depend on the function h that is applied in the soft threshold window. In the example of exponential noise the situation simplifies even further. Our calculations show that the Fisher information, as a function of the noise variance, exhibits stochastic resonance for various noise distributions. In the case of noise with compact support, we sketch a proof that stochastic resonance occurs.

In Section 4, we show how the information transmission function depends on the width of the soft threshold window, including the case where the window is open on one side. We see that when the signal slips inside the window, stochastic resonance disappears.

Section 5 explores periodic signals and the comparison of Fisher information to signal-to-noise ratio for such signals. We sketch an argument showing that stochastic resonance arises in the numerator (the Fourier coefficient of the output signal) of the signal-to-noise ratio. A further discussion of the results concludes Section 5.

2 Fisher information and invertible transfer functions

In previous work on soft thresholds and stochastic resonance a variety of mathematical transfer functions, T, have been studied. To our knowledge, all have been invertible functions, that is one-to-one functions. The logistic function (e.g. Chapeau–Blondeau and Godivier, 1997), the exponential (e.g. Bezrukov and Vodyanoy, 1997) and the cubic (e.g. Vilar et al., 1998) have all been shown to yield stochastic resonance using a signal-to-noise ratio (SNR) measure of information transmission about a periodic signal. The situation is quite different if Fisher information about, e.g. a constant signal, s, is used as a measure of information transmission across a soft threshold. Fisher information, the inverse of the asymptotic minimum variance of any regular estimator of the signal, is known to be invariant with respect to invertible transformations (e.g. Bickel et al., 1998). This property is desirable since one certainly does not wish to measure a change of information if the noisy input signal can be regained by simply inverting the data. This implies that a model using a logistic transfer function produces the same Fisher information as if T were absent. There can be no stochastic resonance in such a model.

More formally, suppose that at each discrete time $t_1, t_2, \ldots, t_n$ the noisy signal $s + \varepsilon(t_i)$ is produced, where the noise variables $\varepsilon(t_i)$ are independent and identically distributed (i.i.d.) with mean zero and variance σ^2. In the following we write briefly ε_i for $\varepsilon(t_i)$. Note that we consider the case of a constant signal function, $s(t) = s$. The probability density function and probability distribution function of ε_i are denoted by f_σ and F_σ respectively. Suppose the soft threshold data $Y_i = Y(t_i)$ are of the form

$$Y_i \quad = \quad T(s + \varepsilon_i)$$

where the transfer function T, also referred to as the "soft threshold function", is an invertible function on the whole real line. If the distribution of ε belongs to a scale family such as the normal distribution, $f_\sigma(z) = f(z/\sigma)/\sigma$, the Fisher information about s is

$$I_s(\sigma) = \int_{-\infty}^{\infty} \frac{f'_\sigma(z)^2}{f_\sigma(z)}\, dz = \frac{1}{\sigma^2} \int_{-\infty}^{\infty} \frac{f'(z)^2}{f(z)}\, dz. \tag{1}$$

This is also the Fisher information for the untransformed signal, $s + \varepsilon_i$. Notice that (1) is a decreasing function of σ. In other words, Fisher information for both untransformed and transformed data always decreases as σ increases from 0, meaning that stochastic resonance does not arise for invertible soft threshold transfer functions.

The question arises, then, why the soft threshold results using SNR, mentioned earlier, were obtained. It is possible that the SNR measure is not invariant to invertible transfer functions.

It is also possible that invertible transfer functions, such as the logistic, can be rendered effectively non-invertible by the limitations of numerical precision of the computations. In the next section, in order to explore this question and to study the transfer function for Fisher information more generally, we define a family of non-invertible soft threshold functions and examine the conditions under which one obtains stochastic resonance. In Section 5 we compare Fisher information and signal-to-noise ratio for a periodic signal.

3 Fisher information and non-invertible transfer functions

We begin by introducing the continuous but non-invertible soft threshold transfer function T. It is convenient to do this in terms of fixed finite numbers, $a < b$, in order to locate the function relative to the signal. As before, we let s be a constant signal that, for now, we think of as being below the threshold a, $s < a$. The noisy signal is $s + \varepsilon_i$, and the distribution and density functions of the noise are f_σ and F_σ. The soft threshold data Y_i are defined by

$$Y_i \;=\; T(s+\varepsilon_i) \;=\; 1_{[b,\infty)}(s+\varepsilon_i) + h(s+\varepsilon_i)1_{(a,b)}(s+\varepsilon_i) \tag{2}$$

where $1_A(x)$ is an indicator function with value 1 if x is in the set A, $x \in A$, and value 0 if not. For example, $1_{[b,\infty)}(s+\varepsilon_i)$ is 1 if $s+\varepsilon_i \geq b$, and zero otherwise. The function h is an invertible function on the restricted range (a,b) only. Notice that T is not invertible over the entire range $(-\infty,+\infty)$. We cannot recover the noisy signal from the soft threshold data, since all values of $s+\varepsilon_i$ below a will have been mapped to 0 and all values above b to 1. The case $a = b$, where T is a step function, was studied by Greenwood et al. (1999, 2003) for both constant and varying signals.

Our assumptions imply that the function h is strictly monotonically increasing on the "soft threshold window" (a,b). If h is linear on (a,b), then T is piecewise linear with slope $1/(b-a)$ for $a < x < b$. Note that transfer functions T as defined in (2) are not only non-invertible but also non-trivial in the sense that they are monotonically increasing with both (non-invertible) constant and (invertible) increasing parts. This does, in particular, exclude (trivial) constant functions which are technically speaking both non-invertible and monotonically increasing (and clearly do not produce stochastic resonance).

In order to compute Fisher information for the soft threshold transfer function we need the probability distribution of Y, which will be denoted by G. We have

$$\begin{aligned} P(Y=0) &= P(s+\varepsilon_i \le a) \;=\; F_\sigma(a-s), \\ P(Y \in dy) &= P(h(s+\varepsilon_i) \in dy) \quad \text{for } \; y \in (0,1), \\ P(Y=1) &= P(s+\varepsilon_i \ge b) \;=\; 1-F_\sigma(b-s). \end{aligned}$$

Hence the distribution G of Y can be written as

$$\begin{aligned} G(dy) &= F_\sigma(a-s)1_{\{0\}}(dy) \\ &\quad + f_\sigma(h^{-1}(y)-s)\frac{1}{h'(h^{-1}(y))}1_{(0,1)}(dy) + (1-F_\sigma(b-s))1_{\{1\}}(dy). \end{aligned} \tag{3}$$

As mentioned in Section 2, Fisher information is the inverse of the minimal asymptotic variance of any regular estimator of s (Bickel et al., 1998). Since our model is parametric, such an *efficient* estimator of s is the maximum likelihood estimator which solves $0 = \sum_{i=1}^n l(Y_i)$, where $l(y) = d/ds \log g(y)$ is the score function and g the probability density of Y. We have $l(0) = -f_\sigma(a-s)/F_\sigma(a-s)$, $l(1) = f_\sigma(b-s)/(1-F_\sigma(b-s))$ and $l(y) = -f'_\sigma(h^{-1}(y)-s)/f_\sigma(h^{-1}(y)-s)$, $0 < y < 1$. Hence the maximum likelihood estimator solves

$$\begin{aligned} 0 &= \sum_{i=1}^n l(Y_i) \\ &= \sum_{i=1}^n \Big(-\frac{f_\sigma(a-s)}{F_\sigma(a-s)}1_{\{0\}}(Y_i) - \frac{f'_\sigma(h^{-1}(Y_i)-s)}{f_\sigma(h^{-1}(Y_i)-s)}1_{(0,1)}(Y_i) + \frac{f_\sigma(a-s)}{1-F_\sigma(b-s)}1_{\{1\}}(Y_i)\Big) \\ &= -\hat{n}_0\frac{f_\sigma(a-s)}{F_\sigma(a-s)} - \sum_{Y_i \in (0,1)} \frac{f'_\sigma(h^{-1}(Y_i)-s)}{f_\sigma(h^{-1}(Y_i)-s)} + \hat{n}_1 \frac{f_\sigma(b-s)}{1-F_\sigma(b-s)} \end{aligned}$$

where $\hat{n}_0$ is the number of i's with $Y_i = 0$ and $\hat{n}_1$ the number of i's with $Y_i = 1$.

The Fisher information about s is the expectation of the squared score function,

$$I_s \;=\; I(s) \;=\; E_s(l(Y))^2.$$

Using $h^{-1}(Y) - s = \varepsilon$, we obtain

$$I_s \;=\; \frac{f_\sigma(a-s)^2}{F_\sigma(a-s)} + \int_{a-s}^{b-s} \frac{f'_\sigma(z)^2}{f_\sigma(z)}\,dz + \frac{f_\sigma(b-s)^2}{1-F_\sigma(b-s)}. \tag{4}$$

When $a = b$, this becomes the Fisher information for a sharp threshold function, or Heaviside function,

$$I_s \;=\; \frac{(f_\sigma(a-s))^2}{F_\sigma(a-s)} + \frac{(f_\sigma(b-s))^2}{1-F_\sigma(b-s)} = \frac{(f_\sigma(a-s))^2}{F_\sigma(a-s)(1-F_\sigma(a-s))}. \tag{5}$$

Notice that the center term of (4) is the Fisher information of the invertible threshold function defined on (a, b), and so is simply omitted in (5) when $a = b$. The left hand term in (4) is the Fisher information for the part of the transfer function below the soft threshold window and the right hand term is that for the part above the window.

We assume that the distribution of ε belongs to a scale family, $f_\sigma(z) = f(z/\sigma)/\sigma$ and $F_\sigma(z) = F(z/\sigma)$. Hence Fisher information as a function of σ is

$$I_s(\sigma) = \frac{1}{\sigma^2}\Big(\frac{(f(\frac{a-s}{\sigma}))^2}{F(\frac{a-s}{\sigma})} + \int_{\frac{a-s}{\sigma}}^{\frac{b-s}{\sigma}} \frac{(f'(z))^2}{f(z)}\,dz + \frac{(f(\frac{b-s}{\sigma}))^2}{1-F(\frac{b-s}{\sigma})}\Big). \tag{6}$$

If $f = \phi$ and $F = \Phi$ are the standard normal density and distribution function, we have

$$I_s(\sigma) = \frac{1}{\sigma^2}\Big(\frac{(\phi(\frac{a-s}{\sigma}))^2}{\Phi(\frac{a-s}{\sigma})} + \int_{\frac{a-s}{\sigma}}^{\frac{b-s}{\sigma}} z^2\phi(z)\,dz + \frac{(\phi(\frac{b-s}{\sigma}))^2}{1-\Phi(\frac{b-s}{\sigma})}\Big). \tag{7}$$

The computations above show that the Fisher information does not depend on h. This same point was made in Section V of Rousseau et al. (2003) for the case where the signal is random and the mutual information between the distributions of input signal and output is used instead of Fisher information.

Exponential noise. If the noise distribution is exponential, i.e. $f(x) = e^{-x}, F(x) = 1 - e^{-x}$, then the Fisher information also does not depend on the upper limit of the soft threshold window. This can be seen easily. Using $F'(x) = f(x) = -f'(x)$ and $f(x) = 1 - F(x)$, the Fisher information is

$$\begin{aligned}
I_s(\sigma) &= \frac{1}{\sigma^2}\Big(\frac{(f(\frac{a-s}{\sigma}))^2}{F(\frac{a-s}{\sigma})} + \int_{\frac{a-s}{\sigma}}^{\frac{b-s}{\sigma}} \frac{(f'(z))^2}{f(z)}\,dz + \frac{(f(\frac{b-s}{\sigma}))^2}{1-F(\frac{b-s}{\sigma})}\Big)\\
&= \frac{1}{\sigma^2}\Big(\frac{(f(\frac{a-s}{\sigma}))^2}{F(\frac{a-s}{\sigma})} + F(\frac{b-s}{\sigma}) - F(\frac{a-s}{\sigma}) + f(\frac{b-s}{\sigma})\Big)\\
&= \frac{1}{\sigma^2}\Big(\frac{(f(\frac{a-s}{\sigma}))^2}{F(\frac{a-s}{\sigma})} - f(\frac{b-s}{\sigma}) + f(\frac{a-s}{\sigma}) + f(\frac{b-s}{\sigma})\Big)\\
&= \frac{1}{\sigma^2}\Big(\frac{(f(\frac{a-s}{\sigma}))^2}{F(\frac{a-s}{\sigma})} + f(\frac{a-s}{\sigma})\Big)
\end{aligned} \tag{8}$$

It does not depend on b. Hence no information is gained if, instead of just the exceedances in the simple sharp threshold model ($a = b$), we observe soft threshold data ($b > a$) or even all data above a threshold ($b = \infty$). The optimal noise level σ, where the curve takes its maximum, is the same in all models.

Compact support. We are interested in the question: which soft thresholded data models exhibit stochastic resonance, i.e. for what models does Fisher information as a function of the noise level σ increase to a maximum and then decrease? This question cannot be answered for all models by a single theoretical argument, even if the distribution F_σ of ε belongs to a scale family, as we assume. If, for example, the noise distribution has compact support, we can verify that stochastic resonance must occur by showing that Fisher information reaches a maximum for some non-zero value of σ, as follows.

Suppose the noise distribution has compact support. This means that all of its probability density lies within a bounded interval. We can show that stochastic resonance is exhibited by verifying that $I_s(\sigma) = 0$ if $\sigma \in [0, a-s)$, $I_s(\sigma) > 0$ if $\sigma \geq a-s$, and $I_s(\sigma) \to 0$ as $\sigma \to \infty$. Suppose, without loss of generality, that the support of the noise density f_σ is $[-\sigma, \sigma]$, i.e. $f_\sigma(z) = f(z/\sigma)/\sigma$ with $f(z) > 0$ on $[-1, 1]$ and zero otherwise. Let $0 \leq \sigma < a - s$. Then $P(Y = 0) = F_\sigma(a-s) = F((a-s)/\sigma) = 1$ and the Fisher information $E(l(Y))^2$ consists only of the first term in (4) pertaining to zero observations, $I_s(\sigma) = f_\sigma(a-s)/F_\sigma(a-s)$. Since $f_\sigma(a-s) = 0$ and $F_\sigma(a-s) = 1$ (in the case $\sigma = 0$, f_σ is replaced by a point mass at zero), we have $I_s(\sigma) = 0$ for $0 \leq \sigma < a - s$.

Let $\sigma \geq a-s$ and consider $I_s(\sigma)$ as given in (4) or (6) with the third summand being possibly zero. Clearly, all terms of the sum are non-negative. Since $f_\sigma(a-s) > 0$ and $F_\sigma(a-s) > 0$, the first summand is strictly greater than zero. This shows $I_s(\sigma) > 0$ if $\sigma \geq a-s$.

To show $I_s(\sigma) \to 0$ for $\sigma \to \infty$, consider the Fisher information (6). The first and third summand tend to a non-negative constant and the second summand decreases to zero as σ increases. Hence $I_s(\sigma) \to 0$ as $\sigma \to \infty$, due to the common factor $1/\sigma^2$.

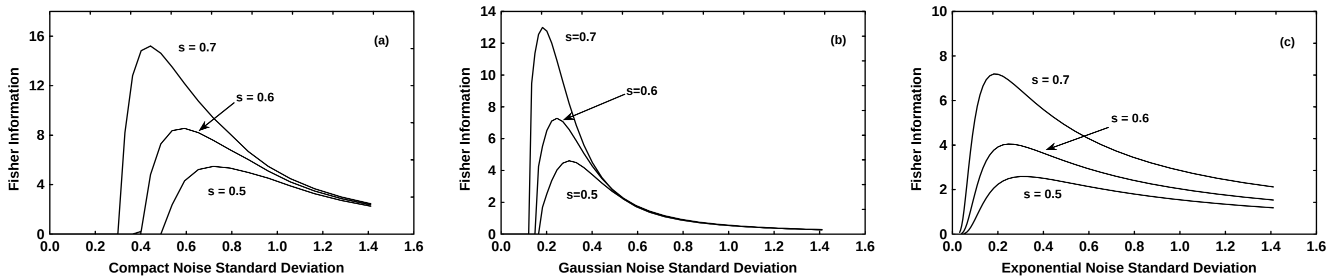

FIG. 1. (a) The Fisher information transmission function for noise density with compact support, $f_\sigma(z) = f(z/\sigma)/\sigma$ with $f(z) = 0.5(\cos(\pi z) + 1)$ for $-1 < z < 1$, soft threshold window $(1, 1.5)$. The integral in the middle term of (6) was evaluated numerically using Neville's algorithm for Romberg integration (Press et al., 1992). (b) The Fisher information transmission function for Gaussian noise, soft threshold window $(1, 1.5)$. The integral in the middle of (7) was evaluated as in (a); the Gaussian distribution was approximated to an accuracy of nine decimal places. (c) The Fisher information transmission function for exponential noise, soft threshold window $(1, 1.5)$.

Figure 1(a) displays the results of a calculation of Fisher information for the compact noise density $f_\sigma(z) = f(z/\sigma)/\sigma$ with $f(z) = 0.5(\cos(\pi z) + 1)$ for $-1 < z < 1$, $f(z) = 0$ otherwise, the soft threshold window $a = 1, b = 1.5$, and three representative values of the signal $s < a$. Stochastic resonance is apparent as predicted by the above analysis.

Remark. If the signal s is random, $s = X + \theta$ where X is a random variable with compact support and θ a location parameter, a similar argument shows that stochastic resonance is exhibited. For both constant and varying signals, transformed by non-trivial non-invertible transfer functions such as those defined in (2), Fisher information displays stochastic resonance.

Because the Gaussian (normal) probability distribution is very similar to one with compact support (there is very little probability density in the tails), we would expect Fisher information to exhibit stochastic resonance for Gaussian noise and the same window. Figure 1(b) displays the results of calculations of Fisher information from (7) for three values of the constant signal s with the soft threshold window $a = 1, b = 1.5$. Stochastic resonance is clearly present. Similar results for exponential noise based on calculations of Fisher information from (8) appear in Figure 1(c). The form of the result does not depend on the invertible portion of the transfer function, h, which does not appear in the equations for Fisher information. In the next section we discuss the effects of changing the width of the soft threshold window and the location of the signal with respect to that window.

4 Changing the soft threshold window

In this section, we study the behavior of the Fisher information and the occurrence of stochastic resonance when the upper and lower bounds of the soft threshold window vary. As mentioned earlier the case $a = b$ corresponds to the sharp threshold model, which was investigated by Greenwood et al. (1999).

Another example of the soft threshold model is the case $b = \infty$, i.e. an invertible transformation h of the data is observed above the threshold a,

$$Y_i \;=\; T(s + \varepsilon_i) \;=\; h(s + \varepsilon_i) 1_{(a,\infty)}(s + \varepsilon_i).$$

Since Fisher information for s will, again, not depend on h, we can take h to be the identity function, i.e. the noisy signal is observed directly if it is large enough. The Fisher information for this model is

$$I(s) \;=\; \frac{(f_\sigma(a - s))^2}{F_\sigma(a - s)} + \int_{a-s}^{\infty} \frac{(f'_\sigma(z))^2}{f_\sigma(z)}\, dz.$$

For $s < a < b$ we expect an increase of information as b increases. In Figure 2 (a), (b) we see, for two values of s, that for $a = 1$ the maximum information does not increase much after

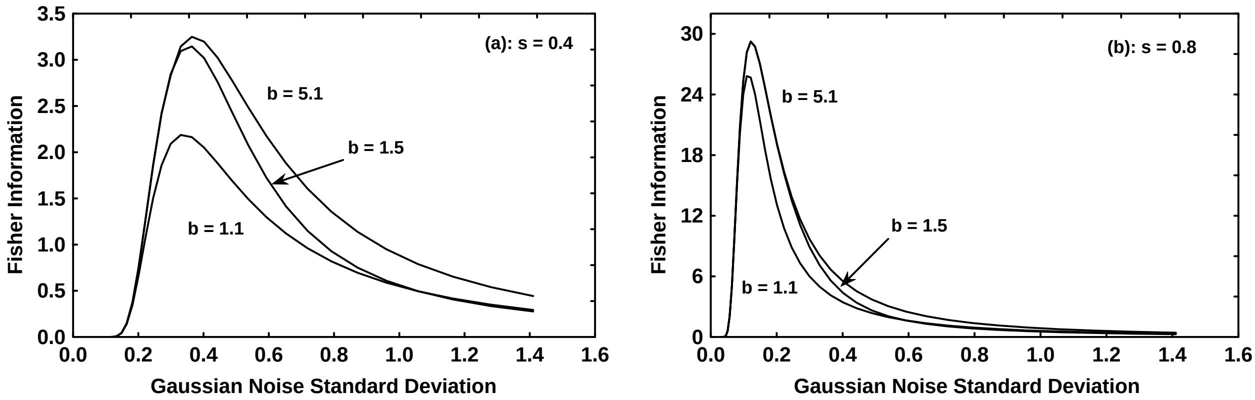

FIG. 2. Fisher information transmission function for Gaussian noise, soft threshold window $(a, b) = (1, 1.1), (1, 1.5)$ and $(1, 5.1)$. (a) The signal $s = 0.4$ is far below the window. (b) The signal $s = 0.8$ is near the lower edge $a = 1$ of the window.

b reaches 1.5. The stochastic resonance point is higher and the peak sharper for s closer to the lower edge of the window.

If $a = -\infty$ and $b = \infty$ then the noisy signal, or an invertible transformation of it, is completely observed, $Y_i = h(s + \varepsilon_i)$. This is the case discussed in Section 2, where Fisher information, $I_s(\sigma)$, computed from (1), is decreasing in σ. An example is the logistic function.

Greenwood et al. (1999) compared Fisher information for the model of a completely observed signal, the sharp threshold model, and the signal observed above a threshold. For example, the proportion of the total available Fisher information for the fully observed signal retained in the sharp threshold model with $s = a = b$ is about 0.64, whereas when $s = a, b = \infty$ the proportion retained is about 0.82, a substantial increase.

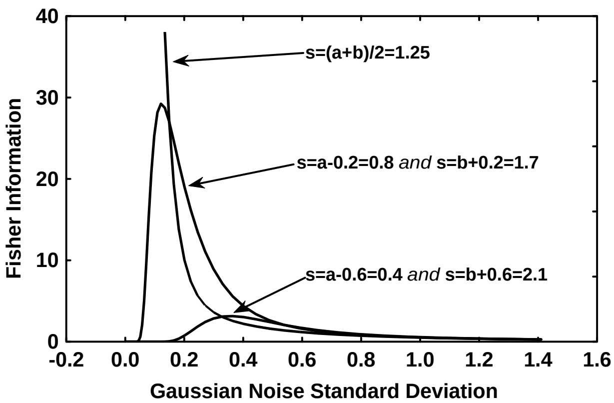

FIG. 3. The Fisher information transmission function shows stochastic resonance when the signal is either above or below the soft threshold window, but not when it is inside the window. The function is symmetric about the soft threshold window, $(1, 1.5)$.

Another question of interest is: do we observe stochastic resonance if instead of the signal being subthreshold, $s < a$, the signal s is inside the soft threshold window, $a < s < b$? The answer is "no" because when s is in the window and $\sigma = 0$, the Y_i give full 'local' information about the location of s. As σ increases, the information can only decrease. This is illustrated in Figure 3 where we see that the case $s = a$ appears to be critical in the sense that for $s < a$, stochastic resonance appears but for $s > a$ it does not. This situation is symmetric with respect to the soft threshold window, with stochastic resonance appearing for both $s < a$ and for $s > b$. The maximum of the information transmission curve increases and occurs at lower noise levels as s approaches the soft threshold window from above or from below.

5 Periodic signal

Much of the literature on stochastic resonance concerns periodic signals and the most-used measure of output signal detectability is the signal-to-noise ratio (SNR). The definition of SNR is, however, not uniform in the literature. If the signal is periodic and the noise is "white", the SNR is the ratio of the amplitude of the signal to the amplitude of the noise. However, if output signal and noise from a specific model are being observed, the SNR is computed in a way appropriate to the model and data at hand. Some examples can be found in Nozaki et al. (1999) and Vilar et al. (1998).

In this section, we compare Fisher information with the SNR as defined by Chapeau–Blondeau and Godivier (1997), who considered soft threshold models with a periodic signal of fixed frequency.

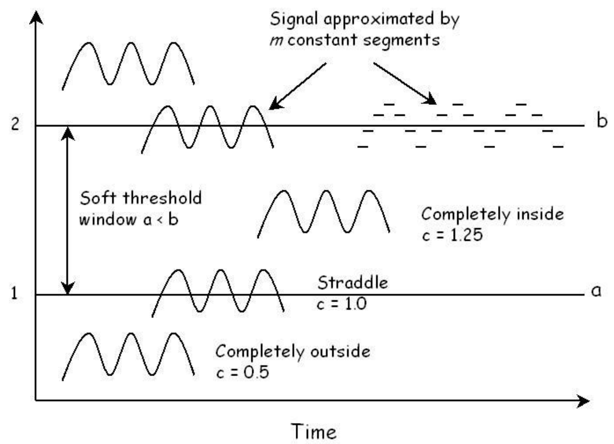

FIG. 4. Possible locations of a periodic signal relative to a soft threshold window, and a periodic signal observed at discrete time points.

Our results for a constant signal extend to the case of a periodic signal with fixed frequency, such as a sinusoid. We center the signal at c and write it as $A\,s(t)+c$, where A is the amplitude and $|s(\cdot)| \leq 1$. Depending on c and A, the signal is either completely outside the soft threshold window (a, b), completely inside the window or partly inside and partly outside the window (Figure 4). The output signal is observed at discrete time points. Hence we can think of the underlying periodic signal as being approximated by a set of m piecewise constant signals (Figure 4). The Fisher information for the signal is the sum of the information calculated for the constant segments. Since each cycle contains a different realization of the noise, we are implicitly averaging over many different realizations. Having defined Fisher information for a periodic signal in a way that permits comparison with the SNR measure, we will compare the two measures as functions of σ.

Consider, for example, a subthreshold periodic signal, $A\,s(t)+c < a < b$. For simplicity we assume that the period is 2π and that we have equally spaced time points at which we observe the output over n periods. Divide each period into m subintervals and denote the endpoints by

$$t_{jk} = j\frac{2\pi}{m} + 2\pi(k-1), \quad j = 1, \ldots, m,\ k = 1, \ldots, n.$$

The observations are $Y_{jk} = T(As(t_{jk}) + c + \varepsilon_{jk})$ where the amplitude $A > 0$ is the parameter and the ε_{jk}'s are i.i.d. errors. Because the periodic signal simply repeats over the n periods, $s(t_{jk}) = s(t_{j1})$. Hence for each j the observations $Y_{j1}, \ldots, Y_{jn}$ are i.i.d. as with a constant signal. As before, we can use a maximum likelihood estimator, a solution of $(d/dA)\{\sum_{k=1}^{n}\sum_{j=1}^{m}\log g(Y_{jk})\} = 0$. The distribution function G and density g are as before (see 3) with s replaced by $As(t_{j1}) + c$. In order to compute Fisher information, one differentiates the log-likelihood $\sum_{k=1}^{n}\sum_{j=1}^{m}\log g(Y_{jk})$ as in Section 3, now with respect to the parameter A. A factor of $s(t_{j1})$ appears as the inner derivative of $As(t_{j1}) + c$ (chain rule). Since the errors are independent, the Fisher information about A is

$$I(A) = \sum_{j=1}^{m} s(t_{j1})^2 I_j(A)$$

with

$$I_j(A) = \frac{f_\sigma(a-(As(t_{j1})+c))^2}{F_\sigma(a-(As(t_{j1})+c))} + \int_{a-(As(t_{j1})+c)}^{b-(As(t_{j1})+c)} \frac{f'_\sigma(z)^2}{f_\sigma(z)}\,dz + \frac{f_\sigma(b-(As(t_{j1})+c))^2}{1-F_\sigma(b-(As(t_{j1})+c))}.$$

Notice that $I_j(A)$ is the Fisher information at the jth time point of a cycle. Since in each cycle the signal values at the jth time point are the same, namely $s(t_{jk}) = s(t_{j1})$, the information $I_j(A)$ is the same as for constant signal (4), except for the value of the signal. The weighted sum of these separate terms gives the Fisher information about A for the entire periodic signal. Notice that the observations where the signal is small or large ($s(\cdot)$ close to 1 or -1) contribute more to the information than those around zero.

From our discussion in Section 4 about constant signals it is clear that for noise distributions with a unimodal density Fisher information, regarded as a function of σ, will behave as follows. If the periodic signal is completely outside the soft threshold window, the information transmission curve is a weighted sum of curves showing stochastic resonance, increasing from zero to a maximum and then decreasing again to zero, thus giving a curve of the same type. If the signal is completely inside the window, there is no stochastic resonance since Fisher information is a weighted sum of decreasing curves, i.e. decreasing in σ. If the signal straddles the upper or lower edge of the window or if it straddles the entire window, we have a mixture of these two types of curves: stochastic resonance and decreasing curves. The shape of the information transmission curve will depend on which type contributes most strongly, i.e. on signal location and amplitude relative to window location and window width.

The corresponding signal-to-noise ratio used by Chapeau–Blondeau and Godivier (1997) at the fundamental frequency is the ratio S/N, where S and N are defined as follows. Consider the first period of the signal. The numerator, S, is the squared absolute value of the Fourier coefficient of the output signal,

$$S \quad = \quad \Big| \frac{1}{m} \sum_{j=1}^{m} EY_{j1} e^{-it_{j1}} \Big|^2 .$$

The denominator, N, is the averaged output variance times the step-width, $2\pi/m$, and a constant bandwidth, ΔB,

$$N \quad = \quad \frac{1}{m} \sum_{j=1}^{m} Var\, Y_{j1} \, \frac{2\pi}{m} \Delta B.$$

We now sketch an argument that the stochastic resonance effect is obtained from the numerator S (the Fourier coefficient of the output signal) of the SNR expression S/R. For simplicity, consider a signal with center 0, $As(t_{j1})$, and a subthreshold signal with one sharp threshold, a. In the denominator, N, the output variance, $Var\, Y$, is an increasing function of the noise standard deviation σ, starting at 0 and then saturating at a certain point when about half of the Y's will be zero and the other half one. This is easily verified for this model, assuming that F is strictly monotonically increasing and symmetric, $F(0) = 1/2$. The variance is $Var\, Y_{j1} = P(Y_{j1} = 1)(1 - P(Y_{j1} = 1)) = (1 - F((a - As(t_{j1}))/\sigma))F((a - As(t_{j1}))/\sigma)$, which is monotonically increasing from $(1 - F(\infty))F(\infty) = 0$ to $(1 - F(0))F(0) = 1/4$ as σ tends from 0 to ∞. Due to this monotonic behavior, the denominator cannot not exhibit stochastic resonance.

That the numerator, S, must have a maximum can be seen as follows. Consider

$$\begin{aligned}
S &= \Big|\frac{1}{m}\sum_{j=1}^{m} EY_{j1}e^{i(-t_{j1})}\Big|^2 \\
&= \Big|\frac{1}{m}\sum_{j=1}^{m} EY_{j1}\Big(\cos(-t_{j1}) + i\sin(-t_{j1})\Big)\Big|^2 \\
&= \frac{1}{m^2}\Big|\sum_{j=1}^{m} EY_{j1}\cos(-t_{j1}) + i\sum_{j=1}^{m} EY_{j1}\sin(-t_{j1})\Big|^2 \\
&= \frac{1}{m^2}\Big((\sum_{j=1}^{m} EY_{j1}\cos(-t_{j1}))^2 + (\sum_{j=1}^{m} EY_{j1}\sin(-t_{j1}))^2\Big). \qquad (9)
\end{aligned}$$

In the sharp threshold case, the j–th expectation is $EY_{j1} = 1 - F((a - As(t_{j1}))/\sigma)$ which is, under the above assumptions and for every j, monotonically increasing from $1 - F(\infty) = 0$ to the constant $1 - F(0) = 1/2$ as σ increases. Hence $S = 0$ for $\sigma = 0$, and $S > 0$ for finite $\sigma > 0$, since the EY_{j1}'s vary with j in this case. For $\sigma \to \infty$ we have $S = 0$: All expectations tend to $1/2$, i.e. S tends to

$$\frac{1}{4m^2}\Big((\sum_{j=1}^{m}\cos(-t_{j1}))^2 + (\sum_{j=1}^{m}\sin(-t_{j1}))^2\Big).$$

Because of the symmetry of the sine and the cosine, and since we assume equally spaced time points, negative and positive values cancel out and the expression is zero.

The same argument shows that $S = 0$ for any value of σ if the signal is constant, $As(t) = s$, since then all expectations are the same, $EY_{j1} = 1 - F((a - s)/\sigma)$ for every j. Hence the SNR degenerates in this case.

One can see, looking at S as computed in (9), that the stochastic resonance behavior of the SNR depends on the behavior in σ of the expectation of Y_{j1}: stochastic resonance results if, for $\sigma = 0$ and for $\sigma \to \infty$, EY_{j1} is the same for all j, whereas for some σ the EY_{j1}'s are not the same for all j. Let us look at the expectations $E(Y_{j1})$ in the soft threshold model (2) with soft threshold window (a, b). For simplicity we assume again that the center, c, of the signal is 0. Then

$$\begin{aligned}
E(Y_{j1}) &= \int T(As(t_{j1}) + \sigma x)f(x)\,dx \\
&= \int_{\frac{a-As(t_{j1})}{\sigma}}^{\frac{b-As(t_{j1})}{\sigma}} h(As(t_{j1}) + \sigma x)f(x)\,dx + 1 - F(\frac{b - As(t_{j1})}{\sigma}). \qquad (10)
\end{aligned}$$

The last two terms correspond to the expectations $E(Y_{j1})$ in the sharp threshold model and exhibit the behavior described in the discussion above. In particular, the expectations increase from 0 to 1/2 as σ increases. We will show that the integral in (10) is 0 for $\sigma = 0$ and $\sigma = \infty$. Since the integral is non-negative, this, combined with the above, will show that the $E(Y_{j1})$'s in the soft threshold model exhibit the same behavior as those in the sharp threshold model, which explains stochastic resonance. Consider, now, the integral in (10). The range of the function h is $[0, 1]$. Hence the integral is bounded from below by 0 and from above by

$$\int_{\frac{a-As(t_{j1})}{\sigma}}^{\frac{b-As(t_{j1})}{\sigma}} f(x)\,dx = F(\frac{b - As(t_{j1})}{\sigma}) - F(\frac{a - As(t_{j1})}{\sigma}).$$

Since the signal is subthreshold, both $b - As(t_{j1})$ and $a - As(t_{j1})$ are non-negative. Hence, for $\sigma = 0$ and $\sigma = \infty$, the integrals are $F(\infty) - F(\infty) = 0$ and $F(0) - F(0) = 0$.

For invertible soft threshold functions analogous considerations apply. Consider the logistic function. An interval, which may be, in effect, a soft threshold window for this case, is an interval around the center of the logistic curve outside of which the function values are approximately zero or one. As $\sigma \to \infty$, the expectations EY_{j1} are 1/2 in the limit. For $\sigma = 0$, these expectations are approximately zero if and only if the signal, i.e. the values $A\,s(t_{j1}) + c$, are outside the approximate soft threshold window.

This argument, showing that stochastic resonance occurs in the numerator, S, of the SNR, may account for the stochastic resonance for subthreshold periodic signals with soft thresholds observed by Chapeau-Blondeau and Godivier (1997) and also by Rousseau et al. (2003).

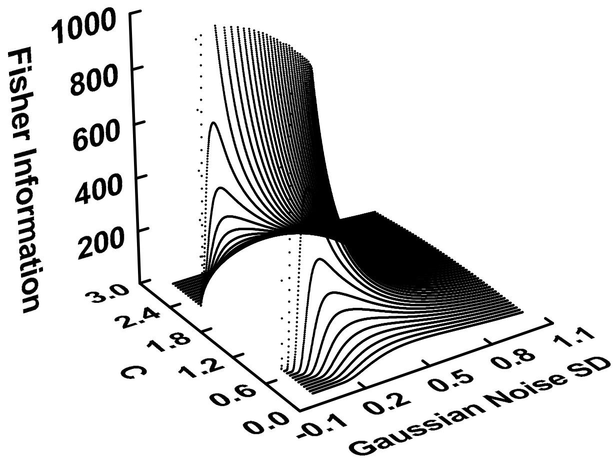

FIG. 5. Fisher information transmission function for a periodic signal centered at c, $0.2\sin(t) + c$. The value of c varies from below the window, $(1, 2)$, to above the window. Stochastic resonance appears for c outside the window. The effect is symmetric with respect to the window. Integrals were evaluated as in Figure 1(b).

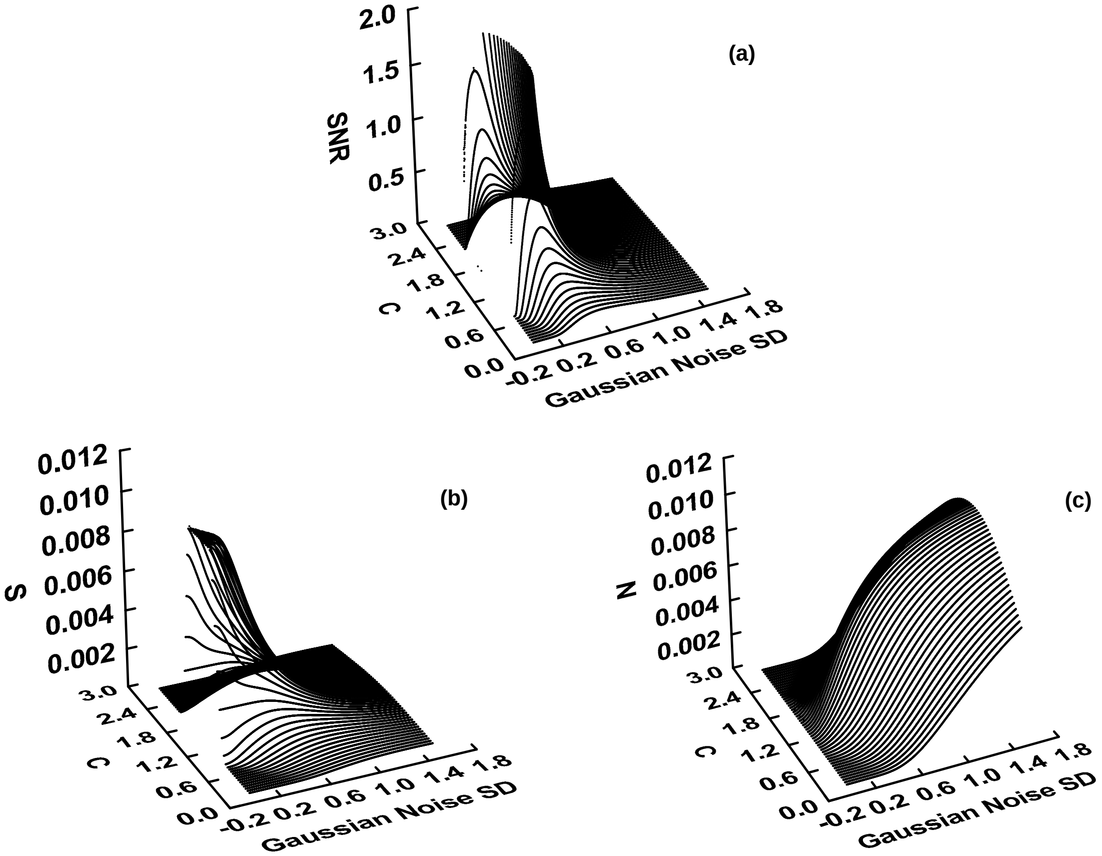

FIG. 6. (a) The SNR information transmission function for a periodic signal centered at c, $0.2\sin(t) + c$, with a linear threshold function h inside the window $(1, 2)$, $m = 128$. Integrals were evaluated as in Figure 1(b). (b) The numerator S (Fourier coefficient of the output signal) of the SNR for the same conditions as in (a). (c) The denominator N (averaged output variance) of the SNR for the same conditions as in (a).

For our illustrations we considered Gaussian noise. In Figures 5, 6 and 7 we used a soft threshold transfer function as in (2) with soft threshold window $(a, b) = (1, 2)$, a sinusoidal signal with amplitude $A = 0.2$, $0.2\sin(t) + c$ and $m = 128$. Figure 5 shows the Fisher information about A whereas Figure 6(a) shows the SNR with linear threshold function h, increasing between $a = 1$ and $b = 2$. Notice that Figures 5 and 6(a) are very similar. The vertical scales are not directly comparable. The Fisher information takes rather large values because we chose a large m. Figures 6(b) and 6(c) show the numerator and denominator of the SNR separately. We see that, as argued above, the numerator shows stochastic resonance when the signal is outside the window. Figures 7(a) and 7(b) show the SNR for h concave upward and downward, respectively, in the window. These figures are related by reverse symmetry. They are similar in form but not identical to Figure 6(a). Although the SNR, unlike Fisher information, does depend on h, the dependence is not very pronounced in our examples.

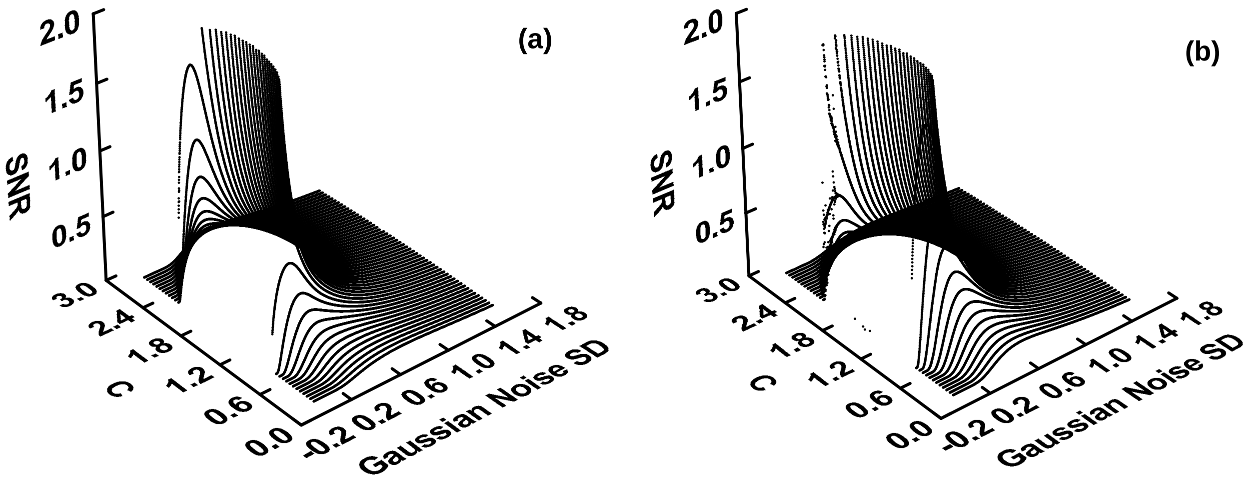

FIG. 7. (a) The SNR information transmission function for a periodic signal centered at c, $0.5\sin(t) + c$, with a concave upward threshold function $h(z) = (z-1)^2$ in the window $(1, 2)$. (b) As (a) but with concave downward threshold function $h(z) = 1 - (z-2)^2$ in the window $(1, 2)$.

Figure 8(a) and (b) show the SNR for a sinusoidal signal with amplitude 0.5 and center c with the logistic function

$$T(z) = h(z) = (1 + e^{-(z-\alpha)/\beta})^{-1}$$

as a soft threshold function. Since this transfer function T is invertible, the Fisher information (not shown), does not exhibit stochastic resonance, as pointed out in Section 2. In Figure 8(a), with $\alpha = 1.5$ and $\beta = 0.05$, the SNR does show stochastic resonance for values of c away from the value of the location parameter α, in line with the above argument concerning the logistic and as found by Chapeau-Blondeau and Godivier (1997). Indeed, for some values of c, the SNR plot closely resembles the function in their Figure 7 for the same value of β, with an initial monotonic drop in SNR at low noise levels followed by a rise in SNR at intermediate noise levels and then a further decline. The behavior may be explained by the denominator of the SNR going to 0 as σ goes to 0, as shown in Figure 6(c). One could examine the relative rates of change in numerator and denominator to establish a firm argument. In Figure 8(b) the logistic function has $\alpha = 1.5$ and $\beta = 0.2$. As in Chapeau-Blondeau and Godivier (1997), we find that the less steep logistic produces no stochastic resonance for some of the same signals (here values of c) for which the steeper logistic in Figure 8(a) did produce stochastic resonance. For values of c farther from the value of α, however, we do see slight stochastic resonance in Figure 8(b). This is again a window-like effect and illustrates the delicate dependence of stochastic resonance on signal placement with respect to the soft threshold function.

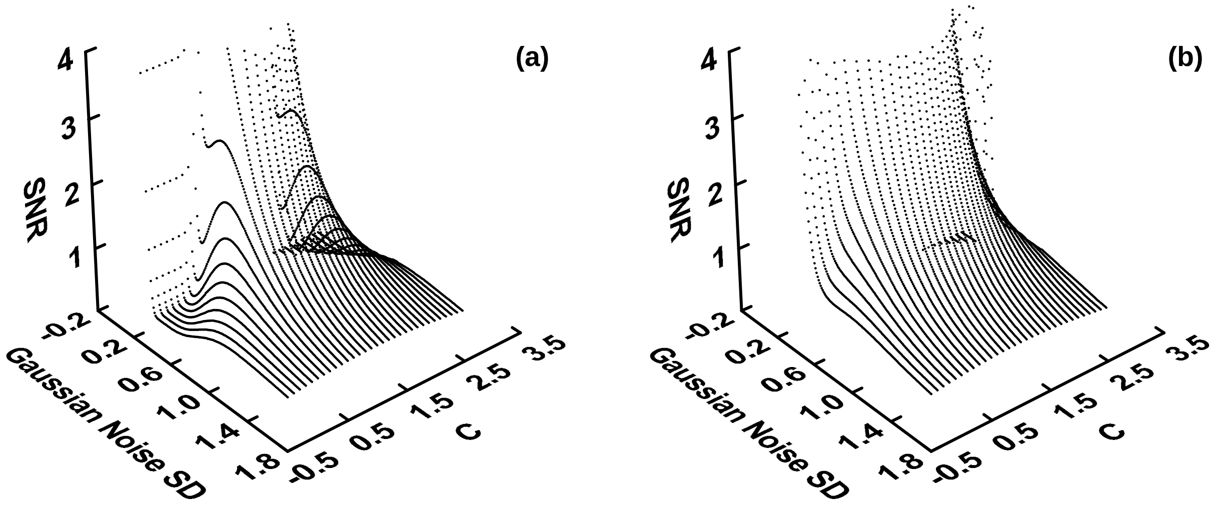

FIG. 8. (a) The SNR information transmission function for a logistic soft threshold transfer function $h(z) = (1 + e^{-(z-\alpha)/\beta})^{-1}$ with $\alpha = 1.5$ and $\beta = 0.05$ and a periodic signal centered at c, $0.5\sin(t) + c$. Integrals were evaluated as in Figure 1(b) except that a generalization of Romberg integration to the case of improper integrals was used. (b) As (a) but with $\alpha = 1.5$ and $\beta = 0.2$.

In summary, Fisher information does not produce stochastic resonance for invertible soft threshold transfer functions but does produce it for non-trivial non-invertible functions and for both constant and changing signals. The SNR information transmission curve is degenerate for constant signals but may exhibit stochastic resonance for periodic signals for both invertible and non-invertible transfer functions. Both measures have the property that stochastic resonance is not found when the signal is entirely within the soft threshold window. For constant signals and Fisher information, the edges of the soft threshold window are critical points for stochastic resonance. For periodic signals, computations show similar results. Since the signal may straddle the edges of the window, the disappearance of stochastic resonance as the signal moves inside the window may be less abrupt.

The Fisher information has theoretical advantages as a measure of information about 'weak' noisy signals moving across soft thresholds. First, it has a unique definition and can be computed for periodic as well as for constant signals. It has an information theoretic status, being the inverse of the asymptotic variance of an efficient estimator of input signal amplitude. It is invariant with respect to invertible transfer functions. The SNR, on the other hand, has been defined in a variety of ways. With the popular definition used here, SNR may exhibit stochastic resonance for invertible transfer functions such as the logistic but its qualitative performance is similar to that of Fisher information. Namely, stochastic resonance appears where the transfer function is relatively constant, growing stronger when the signal is near an interval of increase, and stochastic resonance disappears in intervals where the transfer function is definitely increasing. What is important for stochastic resonance, whether measured by Fisher information or SNR, is the signal being near an edge between an interval of relatively constant transfer function and an interval where this function increases. The slope of the transfer function is not, in itself, of great importance.

The difference between the two measures may not be of great practical importance. In all living organisms, and many other natural systems, there are energy or information quantities below which the system simply does not respond and other quantities above which the response saturates, creating a non-invertible non-linearity that will result in stochastic resonance under the conditions we have indicated, using either measure of information transfer.

Acknowledgments. This research was supported by a grant from the Natural Sciences and Engineering Research Council (NSERC) of Canada to P.E. Greenwood and by a Discovery Grant from NSERC to L.M. Ward. P.E.G. thanks the Center for Evolutionary Functional Genomics at Arizona State University for support. We thank a referee for helpful comments.

References

[1] V.S. Anishchenko, V. Astakhov, A.B. Neiman, T. Vadivasova, and L. Schimansky-Geier. *Nonlinear Dynamics of Chaotic and Stochastic Systems. Tutorial and Modern Developments.* Springer, New York, 2002.

[2] R. Benzi, A. Sutera, and A. Vulpiani. The mechanism of stochastic resonance. *J. Phys. A: Math. Gen.*, **14**, L453–L457, 1981.

[3] S.M. Bezrukov, and I. Vodyanoy. Stochastic resonance without a threshold. *Nature*, **385**, 319-321, 1997.

[4] P.J. Bickel, C.A.J. Klaassen, Y. Ritov, and J.A. Wellner. *Efficient and Adaptive Estimation for Semiparametric Models.* Springer, New York, 1998.

[5] F. Chapeau-Blondeau and X. Godivier. Theory of stochastic resonance in signal transmission by static nonlinear systems. *Phys. Rev. E*, **55**, 1478–1495, 1997.

[6] J.J. Collins, T.T. Imhof, and P. Grigg. Noise-enhanced information transmission in rat sa1 cutaneous mechanoreceptors via aperiodic stochastic resonance. *J. Neurophysiology*, **76**, 64–645, 1996.

[7] L. Gammaitoni, P. Hänggi, P. Jung, and F. Marchesoni. Stochastic resonance. *Rev. Modern Phys.*, **70**, 223–288, 1998.

[8] F.A. Geldard. *The Human Senses.* John Wiley and Sons, New York,1972.

[9] B.J. Gluckman, T.I. Netoff, E.J. Neel, W.L. Ditto, M.L. Spano, and S.J. Schiff. Stochastic resonance in a neuronal network from mammalian brain. *Phys. Rev. Lett.*, **77**, 4098–4101, 1996.

[10] P.E. Greenwood, U.U. Müller, L.M. Ward, and W. Wefelmeyer. Statistical analysis of stochastic resonance in a thresholded detector. *Austrian J. Stat.*, **32**, 1 & 2, 49–70, 2003.

[11] P.E. Greenwood, L.M. Ward, and W. Wefelmeyer. Statistical analysis of stochastic resonance in a simple setting. *Phys. Rev. E*, **60**, 4687–4695, 1999.

[12] A. Longtin. Stochastic resonance in neuron models. *J. Stat. Phys.*, **70**, 309–327, 1993.

[13] A. Longtin. Autonomous stochastic resonance in bursting neurons. *Phys. Rev. E*, **55**, 868–876, 1997.

[14] N.A. Macmillan and C.D. Creelman. *Detection Theory: A User's Guide.* Cambridge University Press, New York, 1990.

[15] U.U. Müller. Nonparametric regression for threshold data. *Canad. J. Statist.*, **28**, 301–310, 2000.

[16] U.U. Müller and L.M. Ward. Stochastic resonance in a statistical model of a time-integrating detector. *Phys. Rev. E (3)*, **61**, 4286–4294, 2000.

[17] D. Nozaki, D.J. Mar, P. Grigg, and J.J. Collins. Effects of colored noise on stochastic resonance in sensory neurons. *Phys. Rev. Lett.*, **82**, 2402–2405, 1999.

[18] W.H. Press, S.A. Teukolsky, W.T. Vetterling, and B.P. Flannery. *Numerical Recipes in C: The Art of Scientific Computing. Second Edition.* Cambridge University Press, New York, 1992.

[19] D. Rousseau, J. Rojas Varela, and F. Chapeau-Blondeau. Stochastic resonance for nonlinear sensors with saturation. *Phys. Rev. E*, **67**, 021102, 1–6, 2003.

[20] M. Stemmler. A single spike suffices: the simplest form of stochastic resonance in model neurons. *Network: Computation in Neural Systems*, **7**, 687–716, 1996.

[21] J.M.G. Vilar, G. Gomila, and J.M. Rubí. Stochastic resonance in noisy nondynamical systems. *Phys. Rev. Lett.*, **81**, 14–17, 1998.

[22] L.M. Ward. *Dynamical Cognitive Science.* MIT Press, Cambridge, MA, 2002.

[23] L.M. Ward. Is stochastic resonance just an epiphenomenon? In S. Bezrukov (ed.) *Unsolved Problems of Noise and Fluctuations*, pp. 84-93. American Physical Society, Washington, D.C., 2003.

[24] L.M. Ward, A. Neiman, and F. Moss. Stochastic resonance in psychophysics and in animal behavior. *Biol. Cybern.*, **87**, 91–101, 2002.

Priscilla E. Greenwood
Department of Mathematics and Statistics
Arizona State University
Tempe, AZ 85287-1804, USA
pgreenw@math.asu.edu
http://math.asu.edu/∼pgreenw/

Ursula U. Müller
Fachbereich 3: Mathematik und Informatik
Universität Bremen
Postfach 330 440
28334 Bremen, Germany
uschi@math.uni-bremen.de
http://www.math.uni-bremen.de/∼uschi/

Lawrence Ward
Department of Psychology
University of British Columbia
Vancouver, BC V6T 1Z4, Canada
lward@psych.ubc.ca
http://www.psych.ubc.ca/∼lward/