

# Estimating the error distribution function in nonparametric regression <sup>\*</sup>

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**Summary:** We construct an efficient estimator for the error distribution function of the nonparametric regression model  $Y = r(Z) + \varepsilon$ . Our estimator is a kernel smoothed empirical distribution function based on residuals from an under-smoothed local quadratic smoother for the regression function.

## 1 Introduction

Consider the nonparametric regression model  $Y = r(Z) + \varepsilon$ , where the covariate  $Z$  and the error  $\varepsilon$  are independent, and  $\varepsilon$  has mean zero, finite variance  $\sigma^2$  and density  $f$ . We observe independent copies  $(Y_1, Z_1), \dots, (Y_n, Z_n)$  of  $(Y, Z)$  and want to estimate the distribution function  $F$  of  $\varepsilon$ . If the regression function  $r$  were known, we could use the empirical distribution function  $\mathbb{F}$  based on the errors  $\varepsilon_1, \dots, \varepsilon_n$ , defined by

$$\mathbb{F}(t) = \frac{1}{n} \sum_{i=1}^n \mathbf{1}\{\varepsilon_i \leq t\}.$$

We consider the regression function as unknown and propose a kernel smoothed empirical distribution function  $\hat{\mathbb{F}}_*$  based on residuals from an under-smoothed local quadratic smoother for the regression function. We give conditions under which  $\hat{\mathbb{F}}_*$  is asymptotically equivalent to  $\mathbb{F}$  plus some correction term:

$$\sup_{t \in \mathbb{R}} n^{1/2} \left| \hat{\mathbb{F}}_*(t) - \mathbb{F}(t) - f(t) \frac{1}{n} \sum_{i=1}^n \varepsilon_i \right| = o_p(1). \quad (1.1)$$

Smoothing the empirical distribution function is appropriate because we assume that the error distribution has a Lipschitz density and therefore a smooth distribution function. A local quadratic smoother for the regression function is appropriate because we assume that the regression function is twice continuously differentiable.

It follows from (1.1) that  $\hat{\mathbb{F}}_*(t)$  has influence function

$$\mathbf{1}\{\varepsilon \leq t\} - F(t) + f(t)\varepsilon.$$

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Müller, Schick and Wefelmeyer (2004a) show that this is the efficient influence function for estimators of  $F(t)$ . Hence  $\hat{\mathbb{F}}_*$  is efficient for  $F$  in the sense that  $(\hat{\mathbb{F}}_*(t_1), \dots, \hat{\mathbb{F}}_*(t_k))$  is a least dispersed regular estimator of  $(F(t_1), \dots, F(t_k))$  for all  $t_1 < \dots < t_k$  and all  $k$ . The influence function of our estimator coincides with the efficient influence function in the model with *constant* regression function; see Bickel, Klaassen, Ritov and Wellner (1998, Section 5.5, Example 1).

It follows in particular from (1.1) that  $\hat{\mathbb{F}}_*(t)$  has asymptotic variance

$$F(t)(1 - F(t)) + \sigma^2 f^2(t) - 2f(t) \int_t^\infty xf(x) dx.$$

If  $f$  is a normal density, this simplifies to

$$F(t)(1 - F(t)) - \sigma^2 f^2(t).$$

Hence, for normal errors, the asymptotic variance of  $\hat{\mathbb{F}}_*(t)$  is strictly smaller than the asymptotic variance  $F(t)(1 - F(t))$  of the empirical estimator  $\mathbb{F}(t)$  based on the *true* errors. This paradox is explained by the fact that the empirical estimator  $\mathbb{F}(t)$  is not efficient: Unlike  $\hat{\mathbb{F}}_*(t)$ , it does not make use of the information that the errors have mean zero. The efficient influence function for estimators of  $F(t)$  from mean zero observations  $\varepsilon_1, \dots, \varepsilon_n$  is

$$\mathbf{1}\{\varepsilon \leq t\} - F(t) - C_0(t)\varepsilon \quad \text{with} \quad C_0(t) = \sigma^{-2} \int_{-\infty}^t xf(x) dx;$$

see Levit (1975). Efficient estimators for  $F(t)$  from observations  $\varepsilon_1, \dots, \varepsilon_n$  are

$$\mathbb{F}(t) - \hat{C}_0(t) \frac{1}{n} \sum_{i=1}^n \varepsilon_i \quad \text{with} \quad \hat{C}_0(t) = \frac{\sum_{i=1}^n \varepsilon_i \mathbf{1}\{\varepsilon_i \leq t\}}{\sum_{i=1}^n \varepsilon_i^2},$$

and the empirical likelihood estimator

$$\frac{1}{n} \sum_{i=1}^n p_i \mathbf{1}\{\varepsilon_i \leq t\}$$

with (random) probabilities  $p_i$  maximizing  $\prod_{i=1}^n p_i$  subject to  $\sum_{i=1}^n p_i \varepsilon_i = 0$ . The empirical likelihood was introduced by Owen (1988), (1990); see also Owen (2001). The asymptotic variance of an efficient estimator  $\mathbb{F}_0(t)$  for  $F(t)$  from  $\varepsilon_1, \dots, \varepsilon_n$  is

$$F(t)(1 - F(t)) - \sigma^{-2} \left( \int_t^\infty xf(x) dx \right)^2.$$

The variance increase of our estimator  $\hat{\mathbb{F}}_*(t)$  over  $\mathbb{F}_0(t)$  is therefore

$$\left( \sigma f(t) - \sigma^{-1} \int_t^\infty xf(x) dx \right)^2.$$

This is the price for not knowing the regression function. For normal errors this term is zero, and we lose nothing. We refer also to the introduction of Müller, Schick and Wefelmeyer (2004b).

Our proof is complicated by two features of the model: the error distribution cannot be estimated adaptively with respect to the regression function, and the regression function cannot be estimated at the efficient rate  $n^{-1/2}$ . Akritas and Van Keilegom (2001) encountered these problems in a related model, the heteroscedastic regression model  $Y = r(Z) + s(Z)\varepsilon$ . They used different techniques and stronger assumptions to get an expansion similar to (1.1). Their results do not cover ours in our simpler model.

Previous related results are easier because at least one of these complicating features is missing. Loynes (1980) assumes that  $Y = h(Z, \vartheta)$ . Koul (1969), (1970), (1987), (1992), Shorack (1984), Shorack and Wellner (1986, Section 4.6) and Bai (1996) consider linear models  $Y = \vartheta^\top Z + \sigma\varepsilon$ . Mammen (1996) studies the linear model as the dimension of  $\vartheta$  increases with  $n$ . Klaassen and Putter (1997) and (2001) construct efficient estimators for the error distribution function in the linear regression model  $Y = \vartheta^\top Z + \varepsilon$ . Koshevnik (1996) treats the nonparametric regression model  $Y = r(Z) + \varepsilon$  with error density symmetric about zero; an efficient estimator for  $F$  is obtained by symmetrizing the empirical distribution function based on residuals. Related results exist for time series. See Boldin (1982), Koul (2002, Chapter 7) and Koul and Leventhal (1989) for linear autoregressive processes  $Y_j = \vartheta Y_{j-1} + \varepsilon_j$ ; Kreiss (1991) and Schick and Wefelmeyer (2002b) for invertible linear processes  $Y_j = \varepsilon_j + \sum_{k=1}^{\infty} \alpha_k(\vartheta)\varepsilon_{j-k}$ ; and Koul (2002, Chapter 8), Schick and Wefelmeyer (2002a) and Müller, Schick and Wefelmeyer (2004c, Section 4) for nonlinear autoregressive processes  $Y_j = r(\vartheta, Y_{j-1}) + \varepsilon_j$ . For invertible linear processes, Schick and Wefelmeyer (2004) show that the smoothed residual-based empirical estimator is asymptotically equivalent to the empirical estimator based on the true innovations. General considerations on empirical processes based on estimated observations are in Ghoudi and Rémillard (1998).

Our result gives efficient estimators  $\int h(t) d\hat{F}_*(t)$  for linear functionals  $E[h(\varepsilon)]$  with bounded  $h$ . For smooth and  $F$ -square-integrable functions  $h$ , it is easier to prove an i.i.d. representation analogous to (1.1) directly; see Müller, Schick and Wefelmeyer (2004a), who also use an under-smoothed estimator for the regression function. Müller, Schick and Wefelmeyer (2004b) compare these results with estimation in the larger model in which one assumes  $E(\varepsilon|Z) = 0$  rather than independence of  $\varepsilon$  and  $Z$  with  $E[\varepsilon] = 0$ . A particularly simple special case is the error variance  $\sigma^2$ , with  $h(x) = x^2$ . For the estimator  $\frac{1}{n} \sum_{i=1}^n \hat{\varepsilon}_i^2$  based on residuals  $\hat{\varepsilon}_i = Y_i - \hat{r}(Z_i)$  with kernel estimator  $\hat{r}$ , under-smoothing is not needed. The asymptotic variance of this estimator was already obtained in Hall and Marron (1990). Müller, Schick and Wefelmeyer (2003) show that a covariate-matched U-statistic is efficient for  $\sigma^2$ ; it does not require estimating  $r$  but uses a kernel density estimator for the covariate density  $g$ . There is a large literature on simpler, inefficient, difference-based estimators for  $\sigma^2$ ; reviews are Carter and Eagleson (1992) and Dette, Munk and Wagner (1998) and (1999).

We can write

$$F(t) = \int \mathbf{1}\{y - r(z) \leq t\} Q(dy, dz),$$

where  $Q$  is the distribution of  $(Y, Z)$ . Our estimator is obtained by plugging in estimators

for  $Q$  and  $r$ . For  $Q$  we use essentially the empirical distribution; for  $r$  we use a local quadratic smoother that is under-smoothed and hence does not have the optimal rate for estimating  $r$ . This means that our estimator does not obey the plug-in principle of Bickel and Ritov (2000) and (2003).

The paper is organized as follows. Section 2 introduces our estimator and states, in Theorem 2.7, the assumptions needed for expansion (1.1). Section 3 derives some consequences of exponential inequalities, and Section 4 contains properties of local polynomial smoothers. Section 5 gives the proof of Proposition 2.8.

## 2 The estimator and the main result

Let us now define our estimator. We begin by defining the residuals. This requires an estimator  $\hat{r}$  of the regression function. We take  $\hat{r}$  to be a local quadratic smoother. To define it we need a kernel  $w$  and a bandwidth  $c_n$ . A local quadratic smoother  $\hat{r}$  of  $r$  is defined as  $\hat{r}(x) = \beta_0(x)$  for  $x \in [0, 1]$ , where  $\beta(x) = (\beta_0(x), \beta_1(x), \beta_2(x))^\top$  is the minimizer of

$$\sum_{j=1}^n \left( Y_j - \beta_0 - \beta_1(Z_j - x) - \beta_2(Z_j - x)^2 \right)^2 \frac{1}{c_n} w\left(\frac{Z_j - x}{c_n}\right).$$

The residuals of the regression estimator  $\hat{r}$  are

$$\hat{\varepsilon}_i = Y_i - \hat{r}(Z_i), \quad i = 1, \dots, n.$$

Let  $\hat{\mathbb{F}}$  denote the empirical distribution function based on these residuals:

$$\hat{\mathbb{F}}(t) = \frac{1}{n} \sum_{i=1}^n \mathbf{1}\{\hat{\varepsilon}_i \leq t\}, \quad t \in \mathbb{R}.$$

Our estimator of the error distribution function will be a smoothed version of  $\hat{\mathbb{F}}$ . To this end, let  $k$  be a density and  $a_n$  another bandwidth. Then we define our estimator  $\hat{\mathbb{F}}_*$  of  $F$  by

$$\hat{\mathbb{F}}_*(t) = \int \hat{\mathbb{F}}(t - a_n x) k(x) dx, \quad t \in \mathbb{R}.$$

With  $K$  the distribution function of  $k$ , we can write

$$\hat{\mathbb{F}}_*(t) = \int K\left(\frac{t - x}{a_n}\right) d\hat{\mathbb{F}}(x) = \frac{1}{n} \sum_{i=1}^n K\left(\frac{t - \hat{\varepsilon}_i}{a_n}\right), \quad t \in \mathbb{R}.$$

This shows that  $\hat{\mathbb{F}}_*$  is the convolution of the empirical distribution function  $\hat{\mathbb{F}}$  of the residuals with the distribution function  $t \mapsto K(t/a_n)$ . Alternatively,  $\hat{\mathbb{F}}_*$  is the distribution function with density  $f_*$  given by

$$f_*(t) = \frac{1}{na_n} \sum_{i=1}^n k\left(\frac{t - \hat{\varepsilon}_i}{a_n}\right), \quad t \in \mathbb{R}.$$

This is the usual kernel density estimator of  $f$  based on the residuals, with kernel  $k$  and bandwidth  $a_n$ . We make the following assumptions.

**Assumption 2.1** *The covariate density  $g$  is bounded and bounded away from zero on  $[0, 1]$ , and its restriction to  $[0, 1]$  is (uniformly) continuous.*

**Assumption 2.2** *The regression function  $r$  is twice continuously differentiable.*

**Assumption 2.3** *The error density  $f$  is Lipschitz, has mean zero, and satisfies the moment condition  $\int |x|^\gamma f(x) dx < \infty$  for some  $\gamma > 4$ .*

**Assumption 2.4** *The density  $k$  is symmetric, twice continuously differentiable, and has compact support  $[-1, 1]$ .*

**Assumption 2.5** *The kernel  $w$  used to define the local quadratic smoother is a symmetric density which has compact support  $[-1, 1]$  and a bounded derivative  $w'$ .*

**Assumption 2.6** *The bandwidths satisfy  $a_n \sim n^{-1/4}/\log n$  and  $c_n \sim n^{-1/4}$ .*

Note that  $c_n$  is smaller than the optimal bandwidth under Assumptions 2.1 and 2.2. Such a bandwidth would be proportional to  $n^{-1/5}$ . This means that our choice of bandwidth results in an under-smoothed local quadratic smoother.

We are now ready to state our main result.

**Theorem 2.7** *Suppose that Assumptions 2.1 to 2.6 hold. Then*

$$\sup_{t \in \mathbb{R}} n^{1/2} \left| \hat{\mathbb{F}}_*(t) - \mathbb{F}(t) - f(t) \frac{1}{n} \sum_{i=1}^n \varepsilon_i \right| = o_p(1).$$

*In particular,  $n^{1/2}(\hat{\mathbb{F}}_* - F)$  converges in distribution in the space  $D([-\infty, \infty])$  to a centered Gaussian process.*

**Proof:** For  $a \in \mathbb{R}$  and  $t \in \mathbb{R}$  set

$$F_a(t) = \int F(t - ax)k(x) dx \quad \text{and} \quad \mathbb{F}_a(t) = \int \mathbb{F}(t - ax)k(x) dx.$$

Since the density  $k$  has mean zero by Assumption 2.4, we have

$$\begin{aligned} F_a(t) - F(t) &= \int (F(t - ax) - F(t) + axf(t))k(x) dx \\ &= \int (-ax) \int_0^1 (f(t - axy) - f(t)) dy k(x) dx. \end{aligned}$$

Thus the Lipschitz continuity of  $f$  yields

$$\sup_{t \in \mathbb{R}} |F_{a_n}(t) - F(t)| = O(a_n^2) = o(n^{-1/2}).$$

It follows from standard empirical process theory that

$$G_n = n^{1/2} \sup_{x \in \mathbb{R}} |\mathbb{F}_{a_n}(x) - F_{a_n}(x) - \mathbb{F}(x) + F(x)| = o_p(1), \quad a_n \rightarrow 0. \quad (2.1)$$

Indeed, with  $W_n = n^{1/2}(\mathbb{F} - F)$  we have

$$G_n = \sup_{t \in \mathbb{R}} \left| \int (W_n(t - a_n s) - W_n(t)) k(s) ds \right| \leq \sup_{t \in \mathbb{R}} \sup_{|s| \leq |a_n|} |W_n(t + s) - W_n(t)|.$$

The above shows that

$$\sup_{t \in \mathbb{R}} n^{1/2} |\mathbb{F}_{a_n}(t) - \mathbb{F}(t)| = o(1).$$

Hence the desired result follows from Proposition 2.8 below.  $\square$

**Proposition 2.8** *Suppose that Assumptions 2.1 to 2.6 hold. Then*

$$\sup_{t \in \mathbb{R}} n^{1/2} \left| \hat{\mathbb{F}}_*(t) - \mathbb{F}_{a_n}(t) - f(t) \frac{1}{n} \sum_{i=1}^n \varepsilon_i \right| = o_p(1).$$

The proof of Proposition 2.8 is in Section 5. We conclude this section with a simple lemma that will be needed repeatedly in the sequel.

**Lemma 2.9** *Suppose that  $\int |x|^\beta dF(x) < \infty$  for some  $\beta > 1$ . Then*

$$\max_{1 \leq i \leq n} |\varepsilon_i| = o_p(n^{1/\beta}).$$

*If  $F$  has also mean zero, then, as  $A \rightarrow \infty$ ,*

$$E[\varepsilon \mathbf{1}\{|\varepsilon| \leq A\}] = o(A^{1-\beta}).$$

**Proof:** The first conclusion follows by the sharper version of the Markov inequality: For  $a > 0$ ,

$$P\left(\max_{1 \leq i \leq n} |\varepsilon_i| > an^{1/\beta}\right) \leq \sum_{i=1}^n P(|\varepsilon_i| > an^{1/\beta}) \leq a^{-\beta} E[|\varepsilon|^\beta \mathbf{1}\{|\varepsilon| > an^{1/\beta}\}] \rightarrow 0.$$

The second conclusion follows from

$$|E[\varepsilon \mathbf{1}\{|\varepsilon| \leq A\}]| = |E[\varepsilon \mathbf{1}\{|\varepsilon| > A\}]| \leq A^{1-\beta} E[|\varepsilon|^\beta \mathbf{1}\{|\varepsilon| > A\}] = o(A^{1-\beta}).$$

In the first equality, we have used that  $\varepsilon$  has mean zero.  $\square$

### 3 Auxiliary Results

In this section we derive some results that will be used in the proof of Proposition 2.8. Let  $(S, \mathfrak{S}, Q)$  be a probability space. For each positive integer  $n$  let  $V, V_1, \dots, V_n$  be independent  $S$ -valued random variables with distribution  $Q$ , and for each  $x$  in  $\mathbb{R}$ , let  $h_{nx}$  be a bounded measurable function from  $S$  into  $\mathbb{R}$ . We first study the process  $H_n$  defined by

$$H_n(x) = \frac{1}{n} \sum_{j=1}^n h_{nx}(V_j) - E[h_{nx}(V)], \quad x \in \mathbb{R}.$$

**Lemma 3.1** *Let  $B_n$  be a sequence of positive numbers such that  $B_n = O(n^\alpha)$  for some  $\alpha > 0$ . Suppose that*

$$\sup_{|x| \leq B_n} \left( E[h_{nx}^2(V)] + \|h_{nx}\|_\infty \right) = O(n/\log n) \quad (3.1)$$

*and, for positive numbers  $\kappa_1$  and  $\kappa_2$ ,*

$$\|h_{ny} - h_{nx}\|_\infty \leq |y - x|^{\kappa_1} O(n^{\kappa_2}), \quad |x|, |y| \leq B_n, \quad |y - x| \leq 1. \quad (3.2)$$

*Then*

$$\sup_{|x| \leq B_n} |H_n(x)| = O_p(1). \quad (3.3)$$

*If we strengthen (3.1) to*

$$\sup_{|x| \leq B_n} \left( E[h_{nx}^2(V)] + \|h_{nx}\|_\infty \right) = o(n/\log n), \quad (3.4)$$

*then*

$$\sup_{|x| \leq B_n} |H_n(x)| = o_p(1). \quad (3.5)$$

**Proof:** To prove the lemma we use an inequality of Hoeffding (1963): If  $\xi_1, \dots, \xi_n$  are independent random variables that have mean zero and variance  $\sigma^2$  and are bounded by  $M$ , then for  $\eta > 0$ ,

$$P\left(\left|\frac{1}{n} \sum_{j=1}^n \xi_j\right| \geq \eta\right) \leq 2 \exp\left(-\frac{n\eta^2}{2\sigma^2 + (2/3)M\eta}\right).$$

Applying this inequality with  $\xi_j = h_{nx}(V_j) - E[h_{nx}(V)]$ , we obtain for  $\eta > 0$ :

$$P(|H_n(x)| \geq \eta) \leq 2 \exp\left(-\frac{n\eta^2}{2E[h_{nx}^2(V)] + 2\eta\|h_{nx}\|_\infty}\right).$$

Thus there is a positive number  $a$  such that for all  $\eta > 0$ ,

$$\sup_{|x| \leq B_n} P(|H_n(x)| \geq \eta) \leq 2 \exp\left(-\frac{\eta^2}{1 \vee \eta} a \log n\right).$$

Now let  $x_{nk} = -B_n + 2kB_n n^{-m}$  for  $k = 0, 1, \dots, n^m$ , with  $m$  an integer greater than  $\alpha + \kappa_2/\kappa_1$ . The above yields for large enough  $\eta > 0$ ,

$$P\left(\max_{k=0, \dots, n^m} |H_n(x_{nk})| > \eta\right) \leq \sum_{k=0}^{n^m} P(|H_n(x_{nk})| > \eta) = o(1).$$

Now, using (3.2),

$$\begin{aligned} \sup_{|x| \leq B_n} |H_n(x)| &\leq \max_{k=0, \dots, n^m} \left( |H_n(x_{nk})| + \sup_{|x-x_{nk}| \leq B_n n^{-m}} |H_n(x) - H_n(x_{nk})| \right) \\ &= O_p(1) + O(B_n^{\kappa_1} n^{-m\kappa_1} n^{\kappa_2}) = O_p(1). \end{aligned}$$

This is the desired result (3.3). The second conclusion is an immediate consequence.  $\square$

Next we consider the degenerate U-process

$$U_n(x) = n^{-m/2} \sum_{(i_1, \dots, i_m) \in I_m^n} u_{nx}(V_{i_1}, \dots, V_{i_m}), \quad x \in \mathbb{R},$$

with  $I_m^n = \{(i_1, \dots, i_m) : 1 \leq i_j \leq n, i_j \neq i_k \text{ if } j \neq k\}$ , and  $u_{nx}$  a bounded measurable function from  $S^m$  to  $\mathbb{R}$  such that for all  $v_1, \dots, v_m$  in  $S$ ,

$$E[u_{nx}(V_1, v_2, \dots, v_m)] = \dots = E[u_{nx}(v_1, v_2, \dots, V_m)] = 0.$$

Set  $\|u_{nx}\|_2 = (E[u_{nx}^2(V_1, \dots, V_m)])^{1/2}$ .

**Lemma 3.2** *Let  $B_n$  be positive numbers such that  $B_n = O(n^\alpha)$  for some  $\alpha > 0$ . Suppose that*

$$\sup_{|x| \leq B_n} \left( \|u_{nx}\|_2^{2/m} + \|u_{nx}\|_\infty^{2/(m+1)} n^{-1/(m+1)} \right) = O((\log n)^{-1}) \quad (3.6)$$

and, for some positive  $\kappa_1$  and  $\kappa_2$ ,

$$\|u_{ny} - u_{nx}\|_\infty \leq |y - x|^{\kappa_1} O(n^{\kappa_2}), \quad |x|, |y| \leq B_n, \quad |y - x| \leq 1. \quad (3.7)$$

Then

$$\sup_{|x| \leq B_n} |U_n(x)| = O_p(1). \quad (3.8)$$

If we strengthen (3.6) to

$$\sup_{|x| \leq B_n} \left( \|u_{nx}\|_2^{2/m} + \|u_{nx}\|_\infty^{2/(m+1)} n^{-1/(m+1)} \right) = o((\log n)^{-1}), \quad (3.9)$$

then

$$\sup_{|x| \leq B_n} |U_n(x)| = o_p(1). \quad (3.10)$$



**Proof:** We use a similar argument as for Lemma 3.1, but rely now on the Arcones–Giné exponential inequality for degenerate U-processes (inequality (c) in Proposition 2.3 of Arcones and Giné, 1994). This inequality states that there are constants  $c_1$  and  $c_2$  depending only on  $m$  such that, for every  $\eta > 0$ , all  $x$  and all  $n$ ,

$$P(|U_n(x)| > \eta) \leq c_1 \exp \left( - \frac{c_2 \eta^{2/m}}{\|u_{nx}\|_2^{2/m} + (\|u_{nx}\|_\infty \eta^{1/m} n^{-1/2})^{2/(m+1)}} \right).$$

From this inequality one obtains as in the proof of Lemma 3.1 that there is a positive number  $b$  such that

$$\sup_{|x| \leq B_n} P(|U_n(x)| > \eta) \leq c_1 \exp \left( - \frac{\eta^{2/m}}{(1 \vee \eta)^{2/(m+m^2)}} b \log n \right), \quad \eta > 0.$$

Now proceed as in the proof of Lemma 3.1.  $\square$

## 4 Properties of local polynomial smoothers

For an introduction to local polynomial smoothers we refer to Fan and Gijbels (1996). In this section we derive some properties of local polynomial smoothers  $\hat{r}$  of order  $d$ , defined by  $\hat{r}(x) = \beta_0(x)$  for  $x \in [0, 1]$ , where  $\beta(x) = (\beta_0(x), \dots, \beta_d(x))^\top$  is the minimizer of

$$\sum_{j=1}^n \left( Y_j - \sum_{m=0}^d \beta_m \left( \frac{Z_j - x}{c_n} \right)^m \right)^2 \frac{1}{c_n} w \left( \frac{Z_j - x}{c_n} \right).$$

Here we have re-scaled  $\beta_1, \dots, \beta_d$  for convenience. The normal equations are

$$Q_n(x) \beta = \frac{1}{n} \sum_{j=1}^n w_n(Z_j - x) Y_j,$$

where the vector  $w_n(x) = (w_{n0}(x), \dots, w_{nd}(x))^\top$  has entries

$$w_{nm}(x) = \frac{x^m}{c_n^{m+1}} w \left( \frac{x}{c_n} \right),$$

and the matrix  $Q_n(x)$  has entries  $q_{n,k+m}(x)$ ,  $k, m = 0, \dots, d$ , with

$$q_{nm}(x) = \frac{1}{n} \sum_{j=1}^n w_{nm}(Z_j - x).$$

By the properties of the kernel  $w$  and the covariate density  $g$  we have for  $m = 0, \dots, 2d$  and all  $x \in \mathbb{R}$ ,

$$|w_{nm}(x)| \leq \|w\|_\infty c_n^{-1}, \quad (4.1)$$

$$|w'_{nm}(x)| \leq (\|w'\|_\infty + m\|w\|_\infty) c_n^{-2}, \quad (4.2)$$

$$E[w_{nm}^2(Z - x)] \leq \|w\|_\infty \|g\|_\infty c_n^{-1}. \quad (4.3)$$

Write  $p_n(x) = (p_{n0}(x), \dots, p_{nd}(x))^\top$  for the first column of the inverse  $Q_n(x)^{-1}$  of  $Q_n(x)$ , and

$$A_n(x, y) = p_n(x)^\top w_n(y - x).$$

From the normal equations we obtain

$$\hat{r}(x) = \beta_0(x) = \frac{1}{n} \sum_{j=1}^n A_n(x, Z_j) Y_j.$$

For the expectation of  $q_{nm}(x)$  we write

$$\bar{q}_{nm}(x) = E[q_{nm}(x)] = \int g(x + c_n t) t^m w(t) dt.$$

We define  $\bar{Q}_n(x)$  correspondingly, replacing  $q_{nm}(x)$  by  $\bar{q}_{nm}(x)$ . Furthermore,  $\bar{p}_n$  and  $\bar{A}_n$  are defined as  $p_n$  and  $A_n$ , with  $Q_n$  replaced by  $\bar{Q}_n$ .

For a unit vector  $v = (v_0, \dots, v_d)^\top$  and  $0 \leq x \leq 1$  we have

$$v^\top \bar{Q}_n(x) v = \int \left( \sum_{i=0}^d v_i t^i \right)^2 g(x + c_n t) w(t) dt.$$

Thus, by Assumption 2.1,

$$(d+1) \|g\|_\infty \geq v^\top \bar{Q}_n(x) v \geq \inf_{0 \leq x \leq 1} g(x) \int_{-1 \vee (-x/c_n)}^{1 \wedge ((1-x)/c_n)} \left( \sum_{i=0}^d v_i t^i \right)^2 w(t) dt.$$

By Assumption 2.5, there is an  $\eta > 0$  such that the eigenvalues of  $\bar{Q}_n(x)$  are in the interval  $[\eta, (d+1) \|g\|_\infty]$  for all  $x$  and  $n$ . Thus  $\bar{Q}_n(x)$  is invertible, and

$$\sup_{0 \leq x \leq 1} \|\bar{Q}_n^{-1}(x)\| \leq 1/\eta. \quad (4.4)$$

**Lemma 4.1** *Suppose Assumptions 2.1 and 2.5 hold. Let  $c_n \rightarrow 0$  and  $c_n^{-1} = O(n/\log n)$ . Then*

$$\sup_{0 \leq x \leq 1} |q_{nm}(x) - \bar{q}_{nm}(x)| = O_p((nc_n/\log n)^{-1/2}), \quad m = 0, \dots, 2d,$$

and consequently

$$\sup_{0 \leq x \leq 1} \|Q_n(x) - \bar{Q}_n(x)\| = O_p((nc_n/\log n)^{-1/2}).$$

**Proof:** Fix  $m$  and use Lemma 3.1 with  $B_n = 1$ ,  $V_j = Z_j$  and

$$h_{nx}(v) = (nc_n/\log n)^{1/2} w_{nm}(v - x).$$

For these choices, the conditions (3.1) and (3.2), with  $\kappa_1 = \kappa_2 = 1$ , follow from (4.1) to (4.3).  $\square$

**Lemma 4.2** Suppose Assumptions 2.1 and 2.5 hold. Assume also that  $f$  has mean zero and finite moment of order  $\beta > 2$ . Let  $c_n \rightarrow 0$  and  $c_n^{-1}n^{2/\beta} = O(n/\log n)$ . Then

$$\sup_{0 \leq x \leq 1} \left| \frac{1}{n} \sum_{j=1}^n w_{nm}(Z_j - x) \varepsilon_j \right| = O_p((nc_n/\log n)^{-1/2}), \quad m = 0, \dots, 2d.$$

**Proof:** Fix  $m$ . In view of Lemmas 2.9 and 4.1 it suffices to show that

$$\sup_{0 \leq x \leq 1} \left| \frac{1}{n} \sum_{j=1}^n w_{nm}(Z_j - x) \varepsilon_{nj} \right| = O_p((nc_n/\log n)^{-1/2}), \quad (4.5)$$

where  $\varepsilon_{nj} = \varepsilon_j \mathbf{1}\{|\varepsilon_j| \leq n^{1/\beta}\} - E[\varepsilon \mathbf{1}\{|\varepsilon| \leq n^{1/\beta}\}]$ . Here we used the fact that

$$(nc_n/\log n)^{1/2} E[\varepsilon \mathbf{1}\{|\varepsilon| \leq n^{1/\beta}\}] = O(n^{-1/2} c_n^{1/2} n^{1/\beta}) = o(1).$$

But (4.5) follows from an application of Lemma 3.1 with  $B_n = 1$ ,  $V_j = (Z_j, \varepsilon_j)$  and

$$h_{nx}(Z_j, \varepsilon_j) = (nc_n/\log n)^{1/2} w_{nm}(Z_j - x) \varepsilon_{nj}.$$

Indeed, the left-hand side of (3.1) is of order  $n/\log n + (nc_n/\log n)^{1/2} c_n^{-1} n^{1/\beta}$ , which is of order  $n/\log n$  by the assumptions on  $c_n$ . Relation (3.2) follows by the Lipschitz continuity of  $w$ .  $\square$

**Theorem 4.3** Suppose Assumptions 2.1 and 2.5 hold. Assume also that  $f$  has mean zero and finite moment of order  $\beta > 2$ . Let  $c_n \rightarrow 0$  and  $c_n^{-1}n^{2/\beta} = O(n/\log n)$ . Then

$$\sup_{0 \leq x \leq 1} \left| \frac{1}{n} \sum_{j=1}^n \bar{A}_n(x, Z_j) \varepsilon_j \right| = O_p((nc_n/\log n)^{-1/2}). \quad (4.6)$$

If, in addition,  $r$  is  $\nu$ -times continuously differentiable with  $\nu \leq d$ , then

$$\sup_{0 \leq x \leq 1} \left| \hat{r}(x) - r(x) - \frac{1}{n} \sum_{j=1}^n \bar{A}_n(x, Z_j) \varepsilon_j \right| = O_p(\log n/(nc_n)) + o_p(c_n^\nu). \quad (4.7)$$

If  $r$  has a Lipschitz continuous  $d$ -th derivative, then  $o_p(c_n^\nu)$  can be replaced by  $O_p(c_n^{d+1})$ .

**Proof:** Since

$$\frac{1}{n} \sum_{j=1}^n \bar{A}_n(x, Z_j) \varepsilon_j = \bar{p}_n(x)^\top \frac{1}{n} \sum_{j=1}^n w_n(Z_j - x) \varepsilon_j,$$

relation (4.6) follows from (4.4) and Lemma 4.2. To prove (4.7), write

$$\hat{r}(x) = \tilde{r}(x) + p_n(x)^\top \frac{1}{n} \sum_{j=1}^n w_n(Z_j - x) \varepsilon_j$$

with

$$\tilde{r}(x) = \frac{1}{n} \sum_{j=1}^n A_n(x, Z_j) r(Z_j).$$

By Lemma 4.1 and relation (4.4),

$$\sup_{0 \leq x \leq 1} \|p_n(x) - \bar{p}_n(x)\| = O_p((nc_n/\log n)^{-1/2}). \quad (4.8)$$

In view of this and Lemma 4.2, assertion (4.7) follows if we verify

$$\sup_{0 \leq x \leq 1} |\tilde{r}(x) - r(x)| = o_p(c_n^\nu). \quad (4.9)$$

By construction,

$$\sum_{j=1}^n A_n(x, Z_j) = 1 \quad \text{and} \quad \sum_{j=1}^n A_n(x, Z_j)(x - Z_j)^m = 0, \quad m = 1, \dots, d.$$

Hence, if we assume that  $r$  is  $\nu$ -times continuously differentiable with  $\nu \leq d$ , we can write

$$\tilde{r}(x) - r(x) = \frac{1}{n} \sum_{j=1}^n A_n(x, Z_j) \left( r(Z_j) - r(x) - \sum_{m=1}^{\nu} r^{(m)}(x) \frac{(Z_j - x)^m}{m!} \right)$$

and obtain the bound

$$|\tilde{r}(x) - r(x)| \leq \frac{1}{n} \sum_{j=1}^n |A_n(x, Z_j)| \frac{c_n^\nu}{\nu!} \sup_{z \in [0,1], |z-x| \leq c_n} |r^{(\nu)}(z) - r^{(\nu)}(x)|.$$

By (4.8), Lemma 4.1 and (4.4),

$$\sup_{0 \leq x \leq 1} \frac{1}{n} \sum_{j=1}^n |A_n(x, Z_j)| = O_p(1). \quad (4.10)$$

The desired (4.9) follows from this and the uniform continuity of  $r^{(\nu)}$  on  $[0, 1]$ . If the  $d$ -th derivative is Lipschitz, one readily sees that (4.9) holds with  $o_p(c_n^\nu)$  replaced by  $O_p(c_n^{d+1})$ .  $\square$

We conclude this section by pointing out an additional property of  $\bar{A}_n$ .

**Lemma 4.4** *Suppose Assumptions 2.1 and 2.5 hold. Let  $c_n \rightarrow 0$  and  $c_n^{-1} = O(n/\log n)$ . Then*

$$\frac{1}{n} \sum_{i=1}^n \left( \frac{1}{n} \sum_{j=1}^n \bar{A}_n(Z_j, Z_i) - 1 \right)^2 = o_p(1). \quad (4.11)$$

**Proof:** Since

$$|\bar{A}_n(z, x)| \leq \sup_{0 \leq y \leq 1} \|\bar{p}_n(y)\| (d+1)^{1/2} \frac{1}{c_n} w\left(\frac{x-z}{c_n}\right),$$

we obtain from Lemma 4.1 and the properties of  $\bar{Q}_n$  that

$$\sup_{0 \leq x \leq 1} \frac{1}{n} \sum_{j=1}^n |\bar{A}_n(x, Z_j)| + \sup_{0 \leq x \leq 1} \frac{1}{n} \sum_{j=1}^n |\bar{A}_n(Z_j, x)| = O_p(1). \quad (4.12)$$

Let  $\Sigma$  be the  $(d+1) \times (d+1)$  matrix with  $(i, j)$ -entry given by  $\int t^{i+j-2} k(t) dt$ . It follows from the uniform continuity of  $g$  on  $[0, 1]$  that

$$\sup_{c_n < x, y < 1-c_n, |x-y| \leq c_n} \|\bar{Q}_n(y) - g(x)\Sigma\| = o(1).$$

This and Lemma 4.1 yield

$$\sup_{c_n < x < 1-c_n} \left| \frac{1}{n} \sum_{j=1}^n w_{nm}(x - Z_j) - g(x) \int (-t)^m k(t) dt \right| = o_p(1)$$

for  $m = 0, \dots, 2d$ . Let  $u$  denote the first column of  $\Sigma$  and  $v$  be the first column of  $\Sigma^{-1}$ . Then we have

$$\sup_{c_n < x, y < 1-c_n, |x-y| \leq c_n} \left\| \bar{p}_n(y) - \frac{1}{g(x)} v \right\| = o(1)$$

and

$$\sup_{c_n < x < 1-c_n} \left\| \frac{1}{n} \sum_{j=1}^n w_n(x - Z_j) - g(x)u \right\| = o(1).$$

Since  $v^\top u = 1$ , we immediately obtain that

$$\sup_{2c_n < x < 1-2c_n} \left| \frac{1}{n} \sum_{j=1}^n \bar{p}_n^\top(Z_j) w_n(x - Z_j) - 1 \right| = o_p(1).$$

The desired result follows from this, (4.12) and the fact that, by Assumption 2.1,

$$\frac{1}{n} \sum_{i=1}^n (\mathbf{1}\{Z_i < 2c_n\} + \mathbf{1}\{Z_i > 1 - 2c_n\}) = o_p(1).$$

□

## 5 Proof of Proposition 2.8

Let  $q_n = (nc_n / \log n)^{-1/2}$ . By choice of  $c_n$  and  $a_n$  we have  $q_n = o(a_n)$ . It follows from Theorem 4.3 that our local quadratic smoother  $\hat{r}$  satisfies

$$\sup_{0 \leq x \leq 1} |\hat{r}(x) - r(x)| = O_p(q_n).$$

This and Lemma 2.9 yield that the probability of the event  $\{\max_{1 \leq j \leq n} |\varepsilon_j| \geq n^{1/3} - a_n\} \cup \{\max_{1 \leq j \leq n} |\hat{\varepsilon}_j| \geq n^{1/3} - a_n\}$  tends to zero. On the complement of this event we have  $\hat{\mathbb{F}}_*(t) = \mathbb{F}_{a_n}(t) = 0$  for all  $t < -n^{1/3}$  and  $\hat{\mathbb{F}}_*(t) = \mathbb{F}_{a_n}(t) = 1$  for all  $t > n^{1/3}$ . Finally,  $\sup_{|t| > n^{1/3}} f(t) = o(1)$  by the uniform continuity of  $f$ . Combining the above and the fact that  $\frac{1}{n} \sum_{j=1}^n \varepsilon_j = O_p(n^{-1/2})$ , we obtain that

$$\sup_{|t| > n^{1/3}} \left| \hat{\mathbb{F}}_*(t) - \mathbb{F}_{a_n}(t) - f(t) \frac{1}{n} \sum_{j=1}^n \varepsilon_j \right| = o_p(n^{-1/2}).$$

Thus we need to show that

$$\sup_{|t| \leq n^{1/3}} \left| \hat{\mathbb{F}}_*(t) - \mathbb{F}_{a_n}(t) - f(t) \frac{1}{n} \sum_{j=1}^n \varepsilon_j \right| = o_p(n^{-1/2}).$$

For this, we first derive some preparatory results. Let  $\phi$  be a Lipschitz-continuous function with compact support contained in  $[-1, 1]$ , and  $b_n$  a sequence of positive numbers such that  $b_n \rightarrow 0$  and  $b_n^{-1} = O(n/\log n)$ . Then it follows from Lemma 3.1 that

$$\sup_{|x| \leq n^{1/3}} \left| \frac{1}{nb_n} \sum_{j=1}^n \phi\left(\frac{x - \varepsilon_j}{b_n}\right) - \int f(x - b_nt) \phi(t) dt \right| = O_p((\log n/(nb_n))^{1/2}). \quad (5.1)$$

Since  $f$  is Lipschitz,

$$\sup_{x \in \mathbb{R}} \left| \int f(x - b_nt) \phi(t) dt - f(x) \int \phi(t) dt \right| = O(b_n). \quad (5.2)$$

It follows from (5.1) and (5.2), with  $\phi$  replaced by  $|\phi|$ , that

$$\sup_{|x| \leq n^{1/3}} \frac{1}{nb_n} \sum_{j=1}^n \left| \phi\left(\frac{x - \varepsilon_j}{b_n}\right) \right| = O_p(1). \quad (5.3)$$

Next, let  $\psi$  be the triangular density defined by  $\psi(x) = (1 - |x|)\mathbf{1}\{|x| \leq 1\}$ . Then we have for  $x \in \mathbb{R}$  and  $u \in \mathbb{R}$  with  $|u| \leq a_n$  that

$$\left| \phi\left(\frac{x - u}{a_n}\right) \right| \leq \|\phi\|_\infty \mathbf{1}\{|x| \leq 2a_n\} \leq 2\|\phi\|_\infty \psi\left(\frac{x}{4a_n}\right).$$

This shows that for all  $t \in \mathbb{R}$  and random variables  $\xi_{n,j}$  and  $\zeta_{n,j}$  we have

$$\left| \frac{1}{n} \sum_{j=1}^n \phi\left(\frac{t - \varepsilon_j + \xi_{n,j}}{a_n}\right) \zeta_{n,j} \mathbf{1}\{|\xi_{n,j}| \leq a_n\} \right| \leq 2\|\phi\|_\infty \frac{1}{n} \sum_{j=1}^n \psi\left(\frac{t - \varepsilon_j}{4a_n}\right) |\zeta_{n,j}|.$$

Thus if  $\max_{1 \leq j \leq n} |\xi_{n,j}| = o_p(a_n)$ , we have

$$\sup_{|t| \leq n^{1/3}} \left| \frac{1}{na_n} \sum_{j=1}^n \phi\left(\frac{t - \varepsilon_j + \xi_{n,j}}{a_n}\right) \zeta_{n,j} \right| = O_p\left(\max_{1 \leq j \leq n} |\zeta_{n,j}|\right). \quad (5.4)$$

We can write

$$\hat{\mathbb{F}}_*(t) = \frac{1}{n} \sum_{j=1}^n K\left(\frac{t - \hat{\varepsilon}_j}{a_n}\right) = \frac{1}{n} \sum_{j=1}^n K\left(\frac{t - \varepsilon_j + \hat{r}(Z_j) - r(Z_j)}{a_n}\right), \quad t \in \mathbb{R}.$$

Choose  $\gamma > 4$  such that  $\int |x|^\gamma f(x) dx < \infty$ . By Lemma 2.9 we have

$$P\left(\max_{1 \leq i \leq n} |\varepsilon_i| > n^{1/\gamma}\right) \rightarrow 0 \quad \text{and} \quad E[\varepsilon_1 \mathbf{1}\{|\varepsilon_1| \leq n^{1/\gamma}\}] = o(n^{-1/2}). \quad (5.5)$$

Let  $\varepsilon_{n,j} = \varepsilon_j \mathbf{1}\{|\varepsilon_j| \leq n^{1/\gamma}\} - E[\varepsilon_j \mathbf{1}\{|\varepsilon_j| \leq n^{1/\gamma}\}]$ . Set

$$\begin{aligned} \delta_{n,i} &= \frac{1}{n} \sum_{j=1}^n \bar{A}_n(Z_i, Z_j) \varepsilon_{n,j}, \quad i = 1, \dots, n, \\ \bar{\mathbb{F}}_*(t) &= \frac{1}{n} \sum_{j=1}^n K\left(\frac{t - \varepsilon_j + \delta_{n,j}}{a_n}\right), \quad t \in \mathbb{R}. \end{aligned}$$

Our next goal is to show that

$$\sup_{|t| \leq n^{1/3}} |\hat{\mathbb{F}}_*(t) - \bar{\mathbb{F}}_*(t)| = o_p(n^{-1/2}). \quad (5.6)$$

It follows from (4.4) and the properties of  $w$  that

$$|\bar{A}_n(x, z)| \leq C_n \mathbf{1}\{|x - z| \leq c_n\}, \quad x, z \in [0, 1], \quad (5.7)$$

where  $C_n = O(c_n^{-1})$ . In view of (4.12) and (5.5), Theorem 4.3 and Lemma 4.4 yield

$$\max_{1 \leq i \leq n} |\hat{r}(Z_i) - r(Z_i) - \delta_{n,i}| = o_p(n^{-1/2}), \quad (5.8)$$

$$\max_{1 \leq i \leq n} |\delta_{n,i}| = O_p(q_n), \quad (5.9)$$

$$\frac{1}{n} \sum_{j=1}^n \delta_{n,j} - \frac{1}{n} \sum_{j=1}^n \varepsilon_j = o_p(n^{-1/2}). \quad (5.10)$$

With  $\zeta_{n,j} = \hat{r}(Z_j) - r(Z_j) - \delta_{n,j}$  we have

$$\hat{\mathbb{F}}_*(t) - \bar{\mathbb{F}}_*(t) = \int_0^1 \frac{1}{na_n} \sum_{j=1}^n \zeta_{n,j} k\left(\frac{t - \varepsilon_j + \delta_{n,j} + s\zeta_{n,j}}{a_n}\right) ds.$$

Using (5.8), (5.9), and (5.4) with  $\phi = k$ , we obtain (5.6).

A Taylor expansion shows that

$$\bar{\mathbb{F}}_*(t) - \mathbb{F}_{a_n}(t) = T_{n,1}(t) + \frac{1}{2} T_{n,2}(t) + \frac{1}{2} R_n(t),$$

where

$$T_{n,1}(t) = \frac{1}{na_n} \sum_{j=1}^n k\left(\frac{t - \varepsilon_j}{a_n}\right) \delta_{n,j}, \quad T_{n,2}(t) = \frac{1}{na_n^2} \sum_{j=1}^n k'\left(\frac{t - \varepsilon_j}{a_n}\right) \delta_{n,j}^2,$$

and

$$R_n(t) = \int_0^1 \frac{1}{na_n^3} \sum_{j=1}^n k''\left(\frac{t - \varepsilon_j + s\delta_{n,j}}{a_n}\right) \delta_{n,j}^3 (1-s)^2 ds.$$

By (5.4) and (5.9),

$$\sup_{|t| \leq n^{1/3}} |R_n(t)| = O_p(a_n^{-2} q_n^3) = o_p(n^{-1/2}).$$

For  $t \in \mathbb{R}$ , let now

$$\begin{aligned} S_{n,1}(t) &= \int f(t - a_n x) k(x) dx \frac{1}{n} \sum_{j=1}^n \delta_{n,j}, \\ S_{n,2}(t) &= a_n^{-1} \int f(t - a_n x) k'(x) dx \frac{1}{n} \sum_{j=1}^n \delta_{n,j}^2. \end{aligned}$$

In view of the Lipschitz continuity of  $f$ , it follows from (5.10) that

$$\sup_{|t| \leq n^{1/3}} \left| S_{n,1}(t) - f(t) \frac{1}{n} \sum_{j=1}^n \varepsilon_j \right| = o_p(n^{-1/2}),$$

and from  $\int k'(x) dx = 0$  that

$$\sup_{|t| \leq n^{1/3}} |S_{n,2}(t)| = O_p\left(\max_{1 \leq i \leq n} \delta_{n,i}^2\right) = O_p(q_n^2) = o_p(n^{-1/2}).$$

Thus the desired result will follow if we show that

$$\sup_{|t| \leq n^{1/3}} |T_{n,\nu}(t) - S_{n,\nu}(t)| = o_p(n^{-1/2}), \quad \nu = 1, 2. \quad (5.11)$$

We shall demonstrate this for the case  $\nu = 2$ . The case  $\nu = 1$  is similar, yet simpler.

We can write  $n^{1/2}(T_{n,2}(t) - S_{n,2}(t)) = \sum_{i=1}^6 U_{n,i}(t)$ , where

$$\begin{aligned} U_{n,1}(t) &= n^{-3/2} \sum_{(i,j,l) \in I_3^n} \phi_{n,t}(\varepsilon_i) n^{-1} a_n^{-2} \bar{A}_n(Z_i, Z_j) \bar{A}_n(Z_i, Z_l) \varepsilon_{n,j} \varepsilon_{n,l}, \\ U_{n,2}(t) &= n^{-1} \sum_{(i,j) \in I_2^n} \phi_{n,t}(\varepsilon_i) n^{-3/2} a_n^{-2} \bar{A}_n^2(Z_i, Z_j) (\varepsilon_{n,j}^2 - E[\varepsilon_{n,1}^2]), \\ U_{n,3}(t) &= n^{-1} \sum_{(i,j) \in I_2^n} \phi_{n,t}(\varepsilon_i) n^{-3/2} a_n^{-2} (\bar{A}_n^2(Z_i, Z_j) - \bar{A}_{ni}^2) E[\varepsilon_{n,1}^2], \\ U_{n,4}(t) &= \frac{1}{n} \sum_{i=1}^n \phi_{n,t}(\varepsilon_i) (n-1) n^{-3/2} a_n^{-2} \bar{A}_{ni}^2 [\varepsilon_{n,1}^2], \\ U_{n,5}(t) &= \frac{2}{n} \sum_{i=1}^n n^{-1/2} a_n^{-2} \tilde{\delta}_{n,i} \phi_{n,t}(\varepsilon_i) \bar{A}_n(Z_i, Z_i) \varepsilon_{n,i}, \\ U_{n,6}(t) &= \frac{1}{n} \sum_{i=1}^n n^{-3/2} \phi_{n,t}(\varepsilon_i) \bar{A}_n^2(Z_i, Z_i) \varepsilon_{n,i}^2, \end{aligned}$$



with

$$\begin{aligned}\phi_{n,t}(\varepsilon_i) &= k'\left(\frac{t - \varepsilon_i}{a_n}\right) - E\left[k'\left(\frac{t - \varepsilon_1}{a_n}\right)\right], \\ \bar{A}_{ni}^2 &= \int \bar{A}_n^2(Z_i, z)g(z) dz, \\ \tilde{\delta}_{n,i} &= \frac{1}{n} \sum_{j:j \neq i} \bar{A}_n^2(Z_i, Z_j)\varepsilon_{n,j}.\end{aligned}$$

Thus we are left to show that

$$\sup_{|t| \leq n^{1/3}} |U_{n,\nu}(t)| = o_p(1), \quad \nu = 1, \dots, 6. \quad (5.12)$$

For  $\nu = 1, 2, 3$  we verify (5.12) with the aid of Lemma 3.2. In each case, (3.7) is a consequence of the Lipschitz continuity of  $k'$ . Thus we only check (3.9).

Note that  $U_{n,1}(t)$  is a degenerate U-statistic of order 3. By (5.7), its kernel  $u_{n,t}$  satisfies  $\sup_{t \in \mathbb{R}} \|u_{n,t}\|_\infty = O(n^{-1}a_n^{-2}c_n^{-2}n^{2/\gamma})$  and  $\sup_{t \in \mathbb{R}} \|u_{n,t}\|_2^2 = O(n^{-2}a_n^{-3}c_n^{-2})$ . Since  $(n^{-2}a_n^{-3}c_n^{-2})^{1/3} + (n^{-1}a_n^{-2}c_n^{-2}n^{2/\gamma})^{1/2}n^{-1/4} = o((\log n)^{-1})$ , we have (3.9) and hence obtain (5.12) for  $\nu = 1$ .

Note that  $U_{n,2}(t)$  is a degenerate U-statistic of order 2. Its kernel  $u_{n,t}$  satisfies  $\sup_{t \in \mathbb{R}} \|u_{n,t}\|_\infty = O(n^{-3/2}a_n^{-2}c_n^{-2}n^{2/\gamma})$  and  $\sup_{t \in \mathbb{R}} \|u_{n,t}\|_2^2 = O(n^{-3}a_n^{-3}c_n^{-3})$ . Thus we have (3.9) and hence (5.12) for  $\nu = 2$ .

Finally,  $U_{n,3}(t)$  is a degenerate U-statistic of order 2. Its kernel  $u_{n,t}$  satisfies

$$\sup_{t \in \mathbb{R}} \|u_{n,t}\|_\infty = O(n^{-3/2}a_n^{-2}c_n^{-2}).$$

Thus we have (3.9) and hence (5.12) for  $\nu = 3$ .

We apply Lemma 3.1 to obtain (5.12) with  $\nu = 4$ . We have (3.4) since its left-hand side is of order  $n^{-1}a_n^{-3}c_n^{-2} + n^{-1/2}a_n^{-2}c_n^{-1}$ . Of course, (3.2) follows since  $k'$  is Lipschitz. Thus we can apply Lemma 3.1 and conclude (5.12) with  $\nu = 4$ .

We obtain from (5.2) and (5.3) with  $\phi = k'$  that

$$\sup_{|t| \leq n^{1/3}} \frac{1}{n} \sum_{i=1}^n |\phi_{n,t}(\varepsilon_i)| = O_p(a_n).$$

Since  $\max_{1 \leq i \leq n} |A_n(Z_i, Z_i)\varepsilon_{n,i}| = O_p(c_n^{-1}n^{1/\gamma})$ , we obtain from (5.9) that

$$\max_{1 \leq i \leq n} |\tilde{\delta}_{n,i}| = O_p(q_n).$$

Thus we have

$$\begin{aligned}\sup_{|t| \leq n^{1/3}} |U_{n,5}(t)| &= O_p(q_n a_n^{-1} c_n^{-1} n^{1/\gamma} n^{-1/2}) = o_p(1) \\ \sup_{|t| \leq n^{1/3}} |U_{n,6}(t)| &= O_p(q_n a_n^{-1} n^{-3/2} c_n^{-2} n^{2/\gamma}) = o_p(1).\end{aligned}$$

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